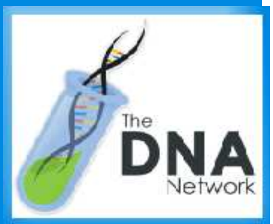

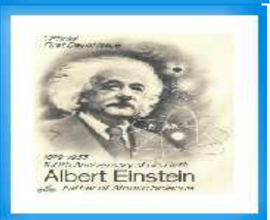

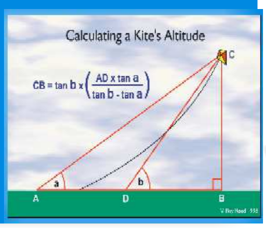

TIFR - MATHEMATICS

MOCK TEST PAPER

- **There are Forty (40) questions divided into four parts (Part A , B, C & D).**
- **Each part consists of 10 True/False type questions.**
- **Each question carry 1 mark and 1 mark will be deducted for each wrong answer.**

- ***Pattern of questions : MCQs***
- ***Total marks : 40***
- ***Duration of test : 2 Hours***

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PART A

1. Two basis of R^3 are $E = \{e_1, e_2, e_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and

$S = \{u_1, u_2, u_3\} = \{(1, 0, 1), (2, 1, 2), (1, 2, 2)\}$

then the change -of-basis matrix P from E to S is $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$.

2. The triangular form of matrix $A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -1 & 2 \\ 0 & -3 & 4 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

3. A function f twice differentiable and satisfying the inequalities $|f(x)| < A$, $|f''(x)| < B$, in the range $x > a$ where A and B are constants then $|f'(x)| < 2\sqrt{AB}$.

4. The invertible matrix of matrix $A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ such that $P^{-1}AP$ is triangular.

5. If $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. and matrix $P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$, then $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

6. The triangular form of matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

7. The sequence $\left\{\frac{n+1}{n}\right\}$ is Unbounded.

8. A function $f : R \to R$ is defined as follows:

$f(x) = 2x^2 + 3x + 4$, if $x \in (-\infty, 1)$ and $f(x) = kx + 9 - k$, if $x \in [1, \infty]$

If this function is differentiable on the whole real line, then the value of k must be 7.

9. Function $\begin{cases} 1, & 0 \le x \le \frac{3\pi}{4} \\ 2\sin\frac{2}{9}x & \frac{3\pi}{4} < x < \pi \end{cases}$ is continuous on $(0, \pi)$.

10. Two basis of R^2 are $S = \{u_1, u_2\} = \{(1, 2), (3, 5)\}$ and $S' = \{v_1, v_2\} = \{(1, -1), (1, -2)\}$, then the change of basis matrix P from S to the "new" basis S' is $\begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$.

PART B

11. E is a non measurable subset of [0, 1]. Let $P = E^{\circ} \cup \{\frac{1}{n} : n \in N\}$ and $Q = \cup\{\frac{1}{n} :\in N\}$ where E^0 is the interior of E and $\bar{E}$ is the closure of E, then P is measurable but not Q.

12. If $a \equiv b \pmod n$. then g.c.d (a, n) = gcd(b, n).

13. The series $\sum\left(\frac{1}{k} + \frac{1}{2^k}\right)$ is convergent.

14. Matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in the Jordan form is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

15. The rational canonical form of matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$. is $\begin{pmatrix} C & & \\ & 1 & & \\ & & 0 & 1 \\ & & -1 & -1 \end{pmatrix}$.

16. The two basis for R^2, B = {(1, –1), (0, 6)} and C = {(2, 1), (–1, 4)} then the transition matrix from C to B is $\begin{bmatrix} 2 & 1/2 \\ -1 & 1/2 \end{bmatrix}$.

17. The standard basis for p_3, B = {1, x, x^2} and the basis C = {p_1, p_2, p_3} where p_1 = 2, p_2 = –4x, p_3 = 5x^2 – 1, then the transition matrix from C to B is $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ -1 & 0 & 5 \end{bmatrix}$.

18. Let T be the linear operative on R^2 defined by T(x, y) = (4x – 2y, 2x + y). then the matrix of T in the basis {f_1=(1, 1), f_2 = (–1, 0)} is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

19. The null space of the matrix $A = \begin{bmatrix} 2 & 0 \\ -4 & 10 \end{bmatrix}$ is {0}.

20. The diagonal form of Matrix $A = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$ is [–1, –6, 5].

PART C

21. If V be the vector space of polynomials t over R of degree ≤ 3, and $D: v \to v$ be the differential operator defined by $D(p(t)) = \frac{d}{dt}(p(t))$. is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

22. The transition matrix P from the basis {e_i} to the basis {f_i} and the transitions matrix q from the basis {f_i} to the basis {e_i} when {e_1 = (1, 0), e_2 = (0, 1)} be a basis of R^2 and {f_1 = (1, 1), f_2 = (–1, 0)} are $p = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}; q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

23. A matrix A over R has $(x - 7)^5$ and $(x - 7)^2$ as its characteristic and minimal polynomial over R respectively. A possible Jordan canonical form is given by)

$$\begin{pmatrix} 7 & 1 & & & \\ & 7 & & & \\ & & 7 & 1 & \\ & & & 7 & \\ & & & & 7 \end{pmatrix}.$$

24. If G is a group and a, $x \in G$, then $O(a) = O(x^{-1}ax)$.

25. The group (R * × R, O), where R * = R – {0} and (a, b) O(c, d) = (ac, bc + d), then the identity element and the inverse of (a, b) are (1, 0) and

$(a^{-1}, -ba^{-1})$, respectively.

26. If G is a group of all 2 × 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $(ab - bc) \neq 0$ and a, b, c, d are integers modulo 3, relative to matrix multiplication, then the number of elements in G is 81.

27. $\lim_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$ equals to 1/e.

28. Let f: G → H be a group homomorphism from a group G into a group H with kernel K. If the order of G, H and K are 75, 45 and 15 respectively, then the order of the image f(G) is 5.

29. In the set Q be rational numbers define $\otimes a$ follows for $\alpha, \beta, \in Q, \alpha \otimes \beta = \frac{\alpha\beta}{3}$

If Q+, Q^{-1}, Q* respectively denote the sets of positive or negative and non-zero rationals, then the pair $(Q^*, \otimes)$ is an abelian group.

30. If S = Z, the set of all integers; $a * b = a + b^2$, then * is binary operation on the given set S.

PART D

31. The equation whose roots are of opposite sign of the equation $x^3 - 6x^2 + 11x - 6 = 0$ is $x^3 + 6x^2 + 11x + 6 = 0$

32. The value of $\lim_{x\to 0}\frac{2^x - 1}{(1+x)^{1/2} - 1}$ is $\log\sqrt{2}$.

33. Sequence $\{a_n\}$, $a_1 > 0$ $a_{n+1} = a_n + \frac{1}{a_n}\ \forall n$ diverges to ∞.

34. $\lim_{x\to a} \sin\frac{1}{x-a} = 0$

35. f(x), g(x) are differential on (a, b) and are continuous on [a, b] and

f(a) = –f(b) = 0 then a point c k (a, b) such that

$f'(c) + f(c)\, g'(c) = 0$

36. There exists a non-abelian group each of whose subgroup is normal.

37. If G is a group of order 10 then it must have a subgroup of order 5.

38. The zero of two multiplicity of $ax^3 + 3bx^2 + 3cx + d$ is $\frac{bc}{2(ac-b)^2}$.

39. There is no element in the ring Z_p which has its own inverse.

40. Let A be a real symmetric matrix and f(x) a polynomial with real coefficient. Then f(A) is also real symmetric.

ANSWER KEY

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Answer	T	F	T	T	T	T	F	T	T	T	T	T	F	T	T
Question	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Answer	F	F	F	T	T	T	F	T	T	T	F	T	T	T	T
Question	31	32	33	34	35	36	37	38	39	40					
Answer	T	F	T	F	T	T	T	F	F	T					

HINTS AND SOLUTION

1. TRUE

Write v = (1, 3, 5) as a linear combination of u_1, u_2, u_3 or equivalent, find $[v]_S$. One way to do this is to directly solve the vector equation $v = xu_1 + yu_2 + zu_3$, that is,

$$\begin{bmatrix}1\\3\\5\end{bmatrix} = x\begin{bmatrix}1\\0\\1\end{bmatrix} + y\begin{bmatrix}2\\1\\2\end{bmatrix} + z\begin{bmatrix}1\\2\\2\end{bmatrix}$$

or $x + 2y + z = 1$

$y + 2z = 3$

$x + 2y + 2z = 5$

Hence the matrix P from E to S is

given by

$$\begin{bmatrix}1 & 2 & 1\\0 & 1 & 2\\1 & 2 & 2\end{bmatrix}$$

2. FALSE

The characteristic polynomial of A is

$$\begin{vmatrix} \lambda-1 & 3 & -3 \\ 0 & \lambda+1 & -2 \\ 0 & 3 & \lambda-4 \end{vmatrix} = (\lambda-1)^2(\lambda-2)$$

Hence the eigenvalues of A are 1, 1, 2. Since the first column of A is already of the required form (with the eigenvalue 1 in the leading place), we process directly to triangularize the submatrix

$$B = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$$

The eigenvalues of B are 1, 2,. To find an eigenvector of B corresponding to the eigenvalue 1, we solve the system of equations

$$(B - I)\, X = 0$$

i.e. $\begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

which yields $x_1 = x_2$. Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of B corresponding to eigenvalue 1. So let

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Then $U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

and $U^{-1}AU = \begin{pmatrix} 1 & -6 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

3. TRUE

Let x> a, and h a positive number; then

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f^n(x+\theta h)$$

or $$hf'(x) = f(x+h) - f(x) - \frac{h^2}{2!} f^n(x+\theta h).$$

$\therefore$ $$|hf'(x)| = |f(x+h) - f(x) - \frac{h^2}{2!} f^n(x+\theta h)|.$$

$$\leq |f(x+h)| + |-f(x)| + \frac{h^2}{2!} | - f^n(x+\theta h)|$$

$$< A + A + \frac{1}{2}h^2 B \qquad \text{[using the gives relations]}$$

or $$|f'(x)| < \frac{2A}{h} + \frac{Bh}{2} = \phi(h), \text{ say}.$$

Now $|f'(x)|$ is independent of h and also less than ϕ (h) for all values of h. Hence $|f'(x)|$ must be less than least value of ϕ (h). For maxima or minima of ϕ (h), we have

$$0 = \phi'(h) = \frac{2A}{h} + \frac{Bh}{2} \text{ or } h = \pm 2\sqrt{\left(\frac{A}{B}\right)}$$

and $\phi''(h) = \dfrac{4A}{h^3} > 0$ when $h = 2\sqrt{\left(\frac{A}{B}\right)}$.

Hence the least value of ϕ (h)

$$= 2A.\ \frac{1}{2}\sqrt{\left(\frac{B}{A}\right)} + \frac{B}{2}.2\sqrt{\left(\frac{A}{B}\right)} = 2\sqrt{(AB)}.$$

Thus. $|f'(x)| < 2\sqrt{(AB)}$.

4. TRUE

Given matrix $A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

The characteristics equation of matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -2 & 0 \\ 1 & -1-\lambda & 2 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(\lambda - 1)^2(\lambda + 1) = 0$$

The eigenvalues of A are 1, 1, –1. It is easily seen that

$X = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ is a solution of $(A - I)X = 0$

hence $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector of A corresponding to eigenvalue 1.

Let us take

$$U = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Since $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent, U is invertible.

By using elementary column operations, it is easily seen that

$$U^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}$$

Hence $U^{-1}AU = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

which is the required triangular form.

5. **TRUE**

The characteristic polynomial of A is

$$f(\lambda) = -\begin{vmatrix} 1-\lambda & 1 & 2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$= \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

Therefore, 1, 2, 3, are the eigenvalues of A. If X_1, X_2, X_3 are eigenvectors corresponding to 1, 2, 3 respectively. Then P = (X'_1, X'_2, X'_3) is the required matrix.

Now $X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ must satisfy

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Therefore a + b + 2c = a, –a + 2b + c = b and b + 3c = c.

These equations give b = –2c, and a = –c. Thus $\alpha\begin{pmatrix}1\\2\\-1\end{pmatrix}$ for any nonzero number α[Here α = –c] is an eigenvector corresponding to 1. Choosing a arbitrarily, say 1, we have $X_1 = \begin{pmatrix}1\\2\\-1\end{pmatrix}$. Similarly,

$$X_2 = \begin{pmatrix}1\\-1\\1\end{pmatrix}, X_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

Therefore $P = \begin{pmatrix}1 & 1 & 1\\2 & -1 & 0\\-1 & 1 & 1\end{pmatrix}$.

then $P^{-1} = \begin{pmatrix}-1 & 0 & 1\\-2 & 2 & 2\\1 & -2 & -3\end{pmatrix}$ and $P^{-1}AP = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix}$.

6. TRUE

$A^2 = \begin{pmatrix}1 & 1 & 0\\-1 & -1 & 0\\0 & 0 & 0\end{pmatrix}$ and $A^3 = 0$. Hence A is nilpotent, with minimum polynomial of A, $q(t) = t^3$. Hence the triangular form of A is the Jordan Canonical form of A which is $\begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}$.

7. FALSE

$$X_n = \frac{n+1}{n} = 1 + \frac{1}{n}$$

$\forall n \in N, 1 \le X_n$ (bounded below)

$\forall n \in N, 2 \geq X_n$ (bounded above)

$\therefore \left\{\frac{n+1}{n}\right\}$ is bounded sequence.

8. TRUE

Here, given functions are

$$f(x) = 2x^2 + 3x + 4, \text{ if } x \in (-\infty, 1)$$

and $f(x) = kx + 9 - k, \text{ if } x \in [1, \infty)$

$\therefore$ $Lf'(x) = Rf'(x)$

Now, $Lf(x) = 2x^2 + 3x + 4$

$\therefore$ $Lf'(x) = 4x + 3$ at $x = 1$

$Lf'(1) = 4 \times 1 + 3 = 7$

Now, $Rf(x) = kx + 9 - k$

$\therefore$ $Rf'(x) = k \Rightarrow Rf'(1) = k$

$\because$ $Lf'(1) = Rf'(1)$

$\Rightarrow$ $\boxed{7 = k}$

9. TRUE

$$f(x) = \begin{cases} 1, & 0 < x \leq \frac{3\pi}{4} \\ 2\sin\left(\frac{2x}{9}\right), & \frac{3\pi}{4} < x < \pi \end{cases}$$

We have, $\lim_{x \to \frac{3\pi}{4}^-} f(x) = 1$

$$\lim_{x \to \frac{3\pi}{4}^+} f(x) = \lim_{x \to \frac{3\pi}{4}} 2\sin\left(\frac{2x}{9}\right) = 1$$

So, f(x) is continuous at $x = \frac{3\pi}{4}$.

$\Rightarrow$ f(x) is continuous at all other points.

10. TRUE

We have

$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ or $\begin{matrix} x + 3y = 1 \\ 2x + 5y = -1 \end{matrix}$ yielding $x = -8,\ y = 3$

$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ or $\begin{matrix} x + 3y = 1 \\ 2x + 5y = -2 \end{matrix}$ yielding $x = -11,\ y = 4$

Thus

$\begin{matrix} v_1 = -8u_1 + 3u_2 \\ v_2 = -11u_1 + 4u_2 \end{matrix}$ and hence $P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$.

Note that the coordinates of v_1 and v_2 are the columns , not rows of the chance-of-basis matrix P.

11. TRUE

E is a non-measurable subset of [0, 1]

Let $P = E^o \cup \left\{\frac{1}{n} : n \in N\right\}$

and $P = \bar{E} \cup \left\{\frac{1}{n} : n \in N\right\}$

Here E^o is the interior of E, i.e. $E^o < E$

and $\bar{E}$ is the closure of E, i.e. $\bar{E} \approx E$

Hence, P is measurable but not Q.

12. TRUE

Let d = g.c.d (a, n)

$\Rightarrow$ d/ | a, d | n, but n | a – b

$\Rightarrow$ d|a – b, d|a

$\Rightarrow$ d|a – (a – b) = b

$\Rightarrow$ d|b, d|n

Let c|b, c|n $\Rightarrow$ c|b, c|a – a as n|a – b

$\Rightarrow$ c|a – b + b = a

$\Rightarrow$ c|a, c|n

$\Rightarrow$ c|d as d = g.c.d(a, n)

$\Rightarrow$ g.c.d(b, n) = d.

13. FALSE

By a well known theorem we know that if Σa_n converges and if Σb_n diverges then $\Sigma(a_n + b_n)$ diverges.

Here the series $\sum \frac{1}{k}$ diverges and $\sum \frac{1}{2^k}$ converges then the series $\sum\left(\frac{1}{k}+\frac{1}{2^k}\right)$ is divergent.

14. TRUE

The characteristic polynomial of A is $(\lambda I - A) = \lambda^4$. Hence $A^4 = 0$, i.e. A is nilpotent. Moreover, by computation $A^3 \neq 0$ so that the minimum polynomial of A is

$q(t) = t^4$. Hence the Jordan canonical form of A is given by

$$\cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

15. TRUE

The characteristic polynomial of A is $(x - 1)\,x(x^2 + x + 1)$ and since the factors are non-square it is also the minimum polynomial of A. Hence the rational canonical form is $C(x) \oplus C(x-1) \oplus C(x^2 + x + 1)$ where $C(q(x))$ is the companion matrix of $q(x)$. In block matrix notation this can be expressed as

$$\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & -1 & -1 \end{pmatrix}$$

16. FALSE

Write the vectors from C as linear combination of the vectors from B. Here are those linear combinations

$$(2, 1) = 2(1, -1) + \frac{1}{2}(0,6)$$

$$(-1, 4) = -(1, -1) + \frac{1}{2}(0,6)$$

The two coordinate matrices are then,

$$[(2,1)]_B = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$$

$$[(-1,4)]_B = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$$

and the transition matrix is given $P = \begin{bmatrix} 2 & -1 \\ 1/2 & 1/2 \end{bmatrix}$

17. FALSE

Since B is the standard basis vectors writing down the transition matrix will be

$$P = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Each column of P will be the coefficients of the vector from C. Since those will also be the coordinate of each of those vectors relative to the standard basis vectors. The first row will be the constant terms from each basis vectors, the second row will be the coefficient of x from each basis vector and third column will be the coefficient of x^2 from each basis vector.

18. FALSE

We have $T(f_1) = T(1, 1) = (2, 3)$

$= 3(1, 1) + (-1, 0)$

$= 3f_1 + f_2$

$T(f_2) = T(-1, 0) = (-4, -2)$

$= 2(1, 1) + 2(-1, 0)$

$= -2f_1 + 2f_2$

Hence $[T]_f = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$

19. TRUE

Since $A = \begin{bmatrix} 2 & 0 \\ -4 & 10 \end{bmatrix}$

to find the null space of A we will need to solve the following system of equations

$$\begin{bmatrix} 2 & 0 \\ -4 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_1 = 0, \qquad -4x_1 + 10x_2 = 0$$

we have given this in both matrix form and equation form. In equation form it is easy to see that the only solution is $x_1 = x_2 = 0$. In terms of vectors form R^2. The solution consists of the single vector {0} and hence the null space of A is {0}.

20. TRUE

The characteristic equation of matrix A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 4-\lambda & 0 & 1 \\ -1 & -6-\lambda & -2 \\ 5 & 0 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4 - \lambda)\,[(6 + \lambda)\lambda] + 1(6 + \lambda)5 = 0$$

$$\Rightarrow (6 + \lambda)\,[\lambda(4 - \lambda) + 5] = 0$$

$$\Rightarrow (6 + \lambda)\,(4\lambda - \lambda^2 + 5) = 0$$

$$\Rightarrow (6 + \lambda)\,(1 + \lambda)\,(5 - \lambda) = 0$$

i.e. $\lambda = -6, -1, 5$

and we know that if we change a matrix in a diagonal form then the diagonal entries are equal to the eigen values of A. Hence the diagonal form of A is (–1, –6, 5).

21. TRUE

$D(1) = 0 = 0 + 0t + 0t^2 + 0t^3$

$D(t) = 1 = 1 + 0t + 0t^2 + 0t^3$

$D(t^2) = 2t = 0 + 2t + 0t^2 + 0t^3$

$D(t^3) = 3t^2 = 0 + 0t + 3t^2 + 0t^3$

22. FALSE

$f_1 = (1, 1) = (1, 0) = (0, 1) = e_1 + e_2$

$f_2 = (-1, 0) = -(1, 0) + 0.(0, 1) = -e_1 + e_2$

Hence the transition matrix P from the basis $\{e_i\}$ to the basis $\{f_i\}$ is

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$e_1 = (1, 0) = 0(1, 1) - (-1, 0) = f_1 - f_2$

$e_2 = (0, 1) = 1(1, 1) + (-1, 0) = f_1 + f_2$

Hence the transition matrix q from the basis $\{f_i\}$

back to the basis $\{e_i\}$ is

$$q = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

Observed that p and q are inverse

$$pq = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = I$$

or By observation : If p and q are change of basis matrix then $p = q^{-1}$ be hold.

23. TRUE

Since the characteristic polynomial of A is $(x - 7)^5$ the characteristic 7 should occur 5 times along the leading diagonal of a possible Jordon-Canonical form J of A. Since $(x - 7)^2$ in the minimal polynomial of A, we must start with a 2 × 2 Jordon black in J i.e., $\begin{matrix} 7 & 1 \\ & 7 \end{matrix}$ in the first Jordan block in J. The Jordan blocks must occur is non increasing order along the principal diagonal.

24. TRUE

Let $a \in G$, $x \in G$, then

$$(x^{-1}ax)^2 = (x^{-1}ax)(x^{-1}ax)$$

$$= x^{-1}(xx^{-1})ax \quad \text{[by associativity]}$$

$$= x^{-1}aeax = x^{-1}(aea)x$$

$$= x^{-1}a^2x$$

Again let $(x^{-1}ax)^{n-1} = x^{-1}a^{n-1}x$, where $(n-1) \in N$

$$\Rightarrow \quad (x^{-1}ax)^{n-1}(x^{-1}ax) = (x^{-1}a^{n-1}x)(x^{-1}ax)$$

$$\Rightarrow \quad (x^{-1}ax)^n = x^{-1}a^{n-1}(xx^{-1})ax = x^{-1}a^{n-1}(eax)$$

Therefore by induction method,

$$(x^{-1}ax)^n = x^{-1}a^nx, \quad \forall n \in N$$

Now let $O(a) = n$ and $O(x^{-1}ax) = m$,

then $(x^{-1}ax)^n = x^{-1}a^nx = x^{-1}ex = e$

$$\Rightarrow \quad O(x^{-1}ax) \le n \quad \Rightarrow m \le n \qquad ...(1)$$

Again $O(x^{-1}ax) = m \Rightarrow (x^{-1}ax)^m = e \Rightarrow x^{-1}a^m x = e$

$\Rightarrow \quad x(x^{-1}a^m x)x^{-1} = xex^{-1} = e$

$\Rightarrow \quad (xx^{-1})a^m(xx^{-1}) = e$

$\Rightarrow \quad ea^m e = e \qquad \Rightarrow \quad a^m = e$

$\Rightarrow O(a) \le m \qquad \Rightarrow n \le m \qquad ...(2)$

(1) and (2) $\Rightarrow \quad n = m \qquad \Rightarrow \quad O(a) = O(x^{-1}ax)$

If O(a) is the infinite, then $O(x^{-1}ax)$ will also be infinite.

25. TRUE

Consider the group $(R^* \times R, \odot)$

where $R^* = R - \{0\}$ and

$(a, b) \odot (c, d) = (ac, bc + d)$ ...(i)

Note the element (a, b) of $(R^* \times R)$

$\Rightarrow a \in R^*, b \in R$

To find identity : Let (c, d) be the identity of $(R^* \times R)$ then

$(a, b) \odot (c, d) = (a, b)$

$\Rightarrow \quad (ac, bc + d) = (a, b)$

$\Rightarrow \quad ac = a$ ...(i)

and $\quad bc + d = b$ ...(ii)

by Eq. (i) $ac = a$

$\Rightarrow \quad ac - a = 0$

$\Rightarrow a(c - 1) = 0.$

since $a \in R^* \Rightarrow a \neq 0$ so, $c = 1$

Now by Eq. (ii) $bc + d = b$

$\Rightarrow \quad bc - b + d = 0$

$\Rightarrow \quad b(c - 1) + d = 0$

$\Rightarrow \quad d = 0 \qquad$ (since, $c = 1$)

Thus identity is $(1, 0) \in (R^*, R)$

Now let (c, d) be inverse of (a, b), then $(c, d) \odot (a, b)$ = identity

$\Rightarrow \quad (c, d) \odot (a, b) = (1, 0)$

$\Rightarrow \quad (ac, bc + d) = (1, 0)$ (by definition of $\odot$)

$\Rightarrow \quad ac = 1 \qquad$ ...(iii)

and $\quad bc + d = 0 \qquad$ ...(iv)

by Eq. (iii) $ac = 1 \Rightarrow c = a^{-1}$

by Eq. (iv) $\quad bc + d = 0$

$\Rightarrow \quad bc = -d$

$\Rightarrow \quad d = -bc$

$= -b\,(a^{-1})$

$= -ba^{-1}$

Hence, identity elements are (1, 0)

and inverse of (a, b) is $(a^{-1}, -ba^{-1})$.

26. FALSE

Here, it is given that G is group of all 2x2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $(ad - bc) \neq 0$ and a,b,c,d are integers modulo 3, So, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then a,b,c,d $\in \{0,1,2\}$ = A (say).

Since a,b,c,d can take three values there are 3 x 3 x 3 = 81, 2 x 2 matrices in act with element in A. If ab = bc = 0, since ab = 0 in five ways (i.e. for five pairs of value of a and b) and bc = 0 in five ways.

There are 5 x 5 = (25) different ways in which ad and bc are simultaneously zero.

If $ad = bc \neq 0$

$ad \neq 0$ means either ad = 1 or 2

Now ad = 1 in two ways and bc = 1 in two ways.

Therefore, both ad and bc are 1 simultaneously in 2 x 2 = (4) ways

Similarly ad = bc = 2 in 4 different ways. Hence, there are eight ways in all in which $ad = bc \neq 0$.

∴ Total number of matrices in the form of ad - bc = 0 is 25 + 8 = (33)

∴ Total number of matrices in the form of

$ad - bc \neq 0$

i.e. 81 - 33 = 48

27. TRUE

Let log p = $\lim_{x \to 0} \frac{\log \operatorname{cosec} x}{\log x} \lim_{x \to 0} \frac{-\cot x}{1/x}$

$$= -\lim_{x\to 0}\frac{x}{\tan x} = -\lim_{x\to 0}\frac{1}{\sec^2 x} = -1$$ (By L' Hospital's rule)

$\Rightarrow$ $p = e^{-1} = 1/e$

28. **TRUE**

Here, It is given that $f : G \to H$ is group homomorphism from a group G into a group H, with kernel K.

$\therefore$ By given condition that O(G) = 75

$$O(H) = 45,\ O(K) = 15$$

$\therefore$ By first fundamental theorem

We have $f(G) \cong \frac{G}{K}$

$$\Rightarrow \quad O\{f(G)\} = O\left(\frac{G}{K}\right)$$

$$\Rightarrow \quad O\{f(G)\} = \frac{O(G)}{O(K)} = \frac{75}{15} = 5$$

29. **TRUE**

Here, it is given that Q is the set of rational numbers which is defined as follows

for a, b $\in$ Q and $\alpha \otimes \beta = \frac{\alpha\beta}{3}$

To prove it as abelian group, it must satisfy the following properties.

1. Closure : Q $\otimes$ is closure in Q*
2. Commutativity : $\alpha \otimes \beta = \frac{\alpha\beta}{3} = \frac{\beta\alpha}{3} = \beta \otimes \alpha$
3. Associativity : $(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)\ \frac{\alpha\beta\gamma}{9},\ \forall\ \alpha, \beta, \gamma \in Q^*$

4. Identity : $\alpha \oplus 3 = \alpha = 3 \otimes \alpha, \forall\ \alpha \in Q^*$

$\therefore$ 3 is identity elements in Q*.

5. Inverse : $\alpha \otimes \frac{9}{\alpha} = 3 = \frac{9}{\alpha} \otimes \alpha, \forall \alpha \in Q^*$

$\therefore\ \frac{9}{\alpha}$ is inverse element

$\therefore (Q,*,\otimes)$ is an abelian group.

30. TRUE

Since, addition is binary operation on the set N of natural numbers i.e, a + b $\in$ N $\forall$ a,b,$\in$ N and subtraction is not a binary operation on N. S = Z, the set of all integers, a * b = a + b^2 satisfy the binary condition.

31. TRUE

The given equation is

$$x^3 - 6x^2 + 11x - 6 = 0$$

replacing x by (–x), the required equation is

$$(-x)^3 - 6(-x)^2 + 11(-x) - 6 = 0$$

or, $-x^3 - 6x^2 - 11x - 6 = 0$

or, $x^3 + 6x^2 + 11x + 6 = 0$

32. FALSE

$$\lim_{x\to 0}\frac{2^x - 1}{(1+x)^{1/2} - 1}$$

$$= \lim_{x\to 0}\frac{2^x \log 2}{\frac{1}{2}(1+x)^{-1/2}} \qquad \left\{\because \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}\right\}$$

= 2log2 = log 4.

33. TRUE

Here $a_n+1 > a_n > 0\ \forall n$. Let the monotonic an be bounded. Then lim $a_n = l(> a1 > 0)$.

On letting $n \to \infty$,

$\ell = \ell + \frac{1}{\ell}$, i.e. $\frac{1}{\ell} = 0$, a contradiction.

Hence, n is unbounded above and being monotonic it diverges to ∞.

34. FALSE

For any d > 0 $\exists n \in \mathbf{N}$ such that

$-\delta < x_1 - a = \frac{1}{-2n\pi - \frac{\pi}{2}} < x_2 - a = \frac{1}{2n\pi + \frac{\pi}{2}} < \delta$, and so,

$x_1, x_2 \in \{x : 0 < |x-a| < \delta\} \Rightarrow \left|\sin\frac{1}{x_1 - a} - \sin\frac{1}{x_2 - a}\right| = |-1-1| = 2 \nless \varepsilon = 1.$

Hence, by the general principle for the existence of limits, $\lim_{x\to a} \sin\frac{1}{x-a}$ does not exist.

35. TRUE

Here $F(x) \equiv f(x)\, e^{g(x)}$ satisfies the conditions of Rolle's theorems on [a, b]. So that there exists a point c $\in$ (a, b) such that F'(c) = 0, i.e.,

$$f(c) = e^{g(c)} + f(c)\, e^{g(c)} = 0.$$

$\therefore$ f'(c) + f(c) g'(c) = 0, as $e^{g(c)} \neq 0$.

36. TRUE

Consider the Quaternion group of order 8.

$$G = \{\pm 1, \pm i, \pm j, \pm k\}, \quad i^2 = j^2 = k^2 = -1$$

and ij = –ji = k, jk = –kj = i, ki = –ik = j

Clearly G is non-abelian. By Lagrange's Theorem, G can have proper subgroups of orders 4 and 2 only. If H is subgroup of G of order 4,

then $i_G(H) = O(G)/O(H) = 8/4 = 2$.

Now $i_G(H) = 2 \Rightarrow H \triangleleft G$.

Thus all subgroups of G of order 4 are normal.

The only subgroup of G of order 2 is {1, –1}, which is obviously normal in G.

Hence G is a non-abelian group each of whose subgroup is normal.

37. TRUE

By Lagrange's theorem such a subgroup can exist. We first claim that all elements of G cannot be order 2. Suppose it is so. Let a, b k G two different elements with order 2.

Let H = <a>, k = <b> be the cyclic subgroups generated by a and b.

then O(H) = 2, O(K) = 2

Since all elements of G are of order 2 it must be abelian.

$\therefore$ HK = KH $\Rightarrow$ HK is a subgroup of G.

and as $O(HK) = \dfrac{O(H)\cdot O(K)}{O(H\cap K)} = \dfrac{2\times 2}{1} = 4$

[Note $H\cap K = (e)$ as $a \neq b$]

By Lagrange's theorem O(HK) should divide O(G) i.e., 4/10 which is not true hence our assumption is wrong and thus all elements of G $\Rightarrow \exists$ can not have order 2.

Again since G is finite O(a)|O(G) for all a $\in$ G at least one element a $\in$ G, such that O(a) = 5 or 10.

If O(a) = 5, then H = <a> is a subgroup of order 5.

If O(a) = 10, then H = $\langle a^2 \rangle$ is a subgroup of order 5.

38. FALSE

Here $F(x) = ax^3 + 3bx^2 + 3cx + d$

$F'(x) = 3ax^2 + 6bx + 3c$

$\therefore$ The zero of F(x) is of multiplicity 1.

Therefore the common divisor of F(x) and F'(x) is

$2(ca - b^2)x + (da - bc)$

which has zero of multiplicity 1. Hence

$$x = \frac{bc - ad}{2(ac - b^2)}$$

39. FALSE

Let $\bar{a} \in Z_p$ such that $\bar{a}^2 = \bar{1}$. Then

$a^2 \equiv 1 (\text{mod}\, p)$ i.e. $p | (a^2 - 1) = (a + 1)(a - 1)$.

$p | a + 1$ or $p | a - 1$, i.e. $a \equiv 1 (\text{mod}\, p)$

or $a \equiv -1 (\text{mod}\, p)$. Hence $\bar{a} = \bar{1}$ or $\bar{a} = \overline{p - 1}$

40. TRUE

Since f has real coefficients

$f(A^t) = f(A)^t$. But $A = A^t$ so that

$f(A) = f(A)^t$. Hence f(A) is symmetric.