

IIT-JAM - MATHEMATICAL STATISTICS

MOCK TEST PAPER(According to new pattern)

- Attempt ALL the 60 questions.
- SECTION-A Consists of 30 questions. These questions are Multiple Choice Questions (MCQs), First 20 questions carries one marks for each and remaining 10 questions carries two marks for each.
- Section-B Consists of 10 questions. These questions are Multiple Select Questions (MSQs), each question carries two marks.
- Section-C Consists of 20 Numerical Answer Type (NAT) questions each question carries two marks. For each NAT type question, the answer is a signed real number.
- In Section A, for all 1 mark questions, 1/3 marks will be deducted for each wrong answer and for all 2 marks questions, 2/3 marks will be deducted for each wrong answer. There is no negative marking in Section B and Section C.

• Total marks : 100

• Duration of test : 3 Hours

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SECTION-A (1-30) : MULTIPLE CHOICE QUESTIONS (MCQs)

1. A coin is tossed $m + n$ times, where $m \geq n$. The probability of getting at least m consecutive heads is
- (A) $\frac{n+1}{2^{m+1}}$
- (B) $\frac{n+2}{2^{m+1}}$
- (C) $\frac{m+2}{2^{n+1}}$
- (D) None of these
2. If a variable takes values $0, 1, 2, \dots, n$ with frequencies $q^n, \frac{n}{1}q^{n-1}p, \frac{n(n-1)}{1 \cdot 2} q^{n-2}p^2, \dots, p^n$, where $p + q = 1$, then the mean is
- (A) np
- (B) nq
- (C) $n(p + q)$
- (D) None of these
3. Let $f(x) = \sin^3 x + \lambda \sin^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum and exactly one maximum.
- (A) $\left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$
- (B) $\left(\frac{-3}{2}, 0\right) \cup \left(\frac{3}{2}, \infty\right)$

(C) $\left(-\infty, -\frac{3}{2}\right) \cup \left(0, \frac{3}{2}\right)$

(D) $\left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$

4. If $a + b + c = 0$, then one of the solution of

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0 \text{ is}$$

- (A) $x = a + b - c$
 (B) $x = 0$
 (C) $x = a - b + c$
 (D) $x = b + c - a$
5. Let $Ax = b$ be a homogeneous system of linear equation. The augmented matrix. $[A : b]$ is given by

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & 0 \end{array} \right]$$

Which of the following statements is true ?

- (A) The system has no Solution
 (B) The system has unique Solution
 (C) The system has infinitely many solutions
 (D) The system has finite number of solutions.

6. A coin is tossed repeatedly until a head appears. Find the expected number of tosses required to obtain the first heads.

(A) $\frac{p}{q}$

(B) $\frac{1}{q}$

(C) $\frac{1}{q^2}$

(D) $\frac{1}{p}$

7. Let X_1, X_2, \dots, X_n be a sequence of mutually independent random variables with common distribution. Suppose X_k assumes only positive integral values and $E(X_k) = a$ exists; $k = 1, 2, \dots, n$. Let $S_n = X_1 + X_2 + \dots + X_n$ then find $E\left(\frac{S_m}{S_n}\right)$ for $1 \leq m \leq n$

(A) $\frac{n}{m}$

(B) $\frac{1}{m}$

(C) $\frac{m}{n}$

(D) $\frac{1}{n}$

8. Let X be a random variables with probability function

$$P(x = \pm 2^x) = \begin{cases} \frac{e^{-1}}{2x!}, & x = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

then find the moment generating function of X.

(A) $M_x(t) = \int_a^\theta \frac{e^{tx}}{x^{\theta+1}} dx$

(B) $M_x(t) = \theta \int_a^\infty \frac{e^{tx}}{x^{\theta+1}} dx$

(C) $M_x(t) = \theta a^\theta \int_a^\theta \frac{e^{tx}}{x^{\theta+1}} dx$

(D) does not exists

9. If X and Y are independent random Variables, and $A = \{x ; a < x < b\}$,

$$B = \{y ; c < y < d\}$$

be any two events, then find $P(A \cap B)$?

(A) $P(A) + P(B)$

(B) $P(A).P(B)$

(C) $P(A)$

(D) $P(B)$

10. Let the joint probability density function of a two dimensional random variable (X, Y) be;

$$f(x, y) = \begin{cases} x+y, & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0, & \end{cases}$$

then find the conditional density function of Y given $X = x$ is

(A) $\frac{2x}{2x+1}; 0 \leq y \leq 1$

(B) $\frac{2(x+y)}{2x+1}; 0 \leq y \leq 1$

(C) $\frac{2y}{2x+1}; 0 \leq y \leq 1$

(D) $\frac{x+y}{2x+1}; 0 \leq y \leq 1$

11. If $f(x) = \begin{cases} ax + b, & \text{if } x \leq \frac{\pi}{4} \\ \cos x, & \text{if } x > \frac{\pi}{4} \end{cases}$ then $f(x)$ is differentiable at $x = \frac{\pi}{4}$, if

(A) $a = -\frac{1}{\sqrt{2}}, b = \frac{1+\frac{\pi}{4}}{\sqrt{2}}$

(B) $a = \frac{1}{\sqrt{2}}, b = \frac{1+\frac{\pi}{4}}{\sqrt{2}}$

(C) $a = b = \frac{1}{\sqrt{2}}$

(D) $a = b = -\frac{1}{\sqrt{2}}$

12. The area of the region enclosed by curves $x^2 + y^2 = a^2, x + y = a$ (in first quadrant).

(A) $a^2(\pi - 2)$

(B) $\frac{a^2}{2}(\pi - 2)$

(C) $\frac{a^2}{3}(\pi - 2)$

(D) $\frac{a^2}{4}(\pi - 2)$.

13. Find the M.L.E of α and β for a random sample from the exponential population

$f(x; \alpha, \beta) = y_0 e^{-\beta(x-\alpha)}$, $\alpha \leq x < \infty$, $\beta \geq 0$ and y_0 being a constant.

(A) $\hat{\alpha} = \bar{x} - x_{(1)}$, $\hat{\beta} = \frac{1}{\bar{x} - x_{(1)}}$

(B) $\hat{\alpha} = \bar{x}$, $\hat{\beta} = \frac{1}{\bar{x} - x_{(1)}}$

(C) $\hat{\alpha} = \frac{1}{\bar{x} - x_{(1)}}$, $\hat{\beta} = x_{(1)}$

(D) $\hat{\alpha} = x_{(1)}$, $\hat{\beta} = \frac{1}{\bar{x} - x_{(1)}}$

14. For the distribution with density function $f(x, \theta) = \frac{x}{1+\theta}$; $0 \leq x \leq 1$ then the estimate of θ by

method of moments

(A) $\frac{1}{3M_1} - 1$

(B) $3M_1 - 1$

(C) $\frac{1}{\sqrt{3M_1 - 1}}$

(D) M_1

15. Find the maximum value of the function $f(x) = x^2 e^{-x}$

(A) $\frac{4}{e^2}$

(B) $\frac{4}{e^3}$

(C) $\frac{2}{e^2}$

(D) $\frac{2}{e^3}$

16. Solve the differential equation

$$y - x \left(\frac{dy}{dx} \right) = a \left[y^2 + \frac{dy}{dx} \right], \quad y(0) = 1$$

(A) $\frac{x+a}{a} = y(1+x)$

(B) $\frac{x+a}{a} = y(x-1)$

(C) $\frac{x-a}{a} = x(1+y)$

(D) $\frac{x-a}{a} = x(y-1)$

17. Suppose $B = \{b_1, b_2\}$ is a basis for V and $C = \{c_1, c_2, c_3\}$ is a basis for W . Let

$T : V \rightarrow W$ be a linear transformation with the property that

$$T(b_1) = 3c_1 - 2c_2 + 5c_3 \quad \text{and} \quad T(b_2) = 4c_1 + 7c_2 - c_3$$

Find the matrix M for T relative to B and C .

(A) $\begin{bmatrix} 3 & -2 & 5 \\ 4 & 7 & -1 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 4 & 7 & -1 \\ 3 & -2 & 5 \end{bmatrix}$

(D) $\begin{bmatrix} 4 & 3 \\ 7 & -2 \\ -1 & 5 \end{bmatrix}$

18. If $X \sim N(0, 1)$ then find the correlation coefficient of (x, y) where $y = a + bx + cx^2$.

(A) $\frac{a}{\sqrt{b^2 + c^2 - ab}}$

(B) $\frac{b}{\sqrt{a^2 + c^2 - ac}}$

(C) $\frac{b}{\sqrt{b^2 + 2c^2 - ac}}$

(D) $\frac{c}{\sqrt{a^2 + 2b^2 - ac}}$

19. Let X_1 and X_2 have the joint pdf

$$f_{x_1, x_2}(x_1, x_2) = \begin{cases} 10x_1x_2^2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

if $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$ then find the marginal pdf of Y_1 .

(A) $f_{y_1}(y_1) = 2y_1, 0 < y_1 < 1$

(B) $f_{y_1}(y_1) = 5y_1^4, 0 < y_1 < 1$

(C) $f_{y_1}(y_1) = \frac{y_1}{2}, 0 < y_1 < 1$

(D) $\frac{y_1}{2} = \frac{y_1^4}{5}, 0 < y_1 < 1$

20. If x be a random variable such that

$$E(x^m) = \frac{(m+3)!}{3!} 3^m, m = 1, 2, 3, \dots$$

and the mgf of x is given by

$$M(t) = 1 + \frac{4!3}{3!1!} t + \frac{5!3^2}{3!2!} t^2 + \frac{6!3^3}{3!3!} t^3 + \dots$$

then the random variable X follows the distribution

- (A) $\gamma(3, 4)$
- (B) $\gamma(4, 3)$
- (C) $\beta(3, 4)$
- (D) $\beta(4, 3)$

21. Let the Random variable x have the p.m.f.

$$f(x, \theta) = \frac{e^{-\theta}\theta^x}{x!}, \quad x = 0, 1, 2, \dots$$

then find the sufficient statistic for θ .

- (A) $\hat{\theta} = \sum_{i=1}^n x_i$
- (B) $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i$
- (C) $\hat{\theta} = \prod_{i=1}^n x_i$
- (D) $\hat{\theta} = \frac{1}{n} \prod_{i=1}^n x_i$

22. A sample of 900 members has a mean 3.4cms. and s.d. 261 cms. Is the sample from a large population of mean 3.25 cms. and s.d. 2.61 cms ?

- (A) Yes
- (B) No
- (C) datas are incomplete

(D) cannot be determined

23. $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t) dt}{x^2 - \frac{\pi^2}{16}}$ equals

(A) $\frac{8}{\pi} f(2)$

(B) $\frac{2}{\pi} f(2)$

(C) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$

(D) $4f(2)$.

24. Three six-faced dice are thrown together. The probability that the sum of the numbers operating on the dice, $k(3 \leq k \leq 8)$, is

(A) $\frac{(k-1)(k-2)}{432}$.

(B) $\frac{(k-1)(k^2-2)}{42}$.

(C) $\frac{(k-3)(k-5)}{432}$.

(D) $\frac{(k-6)(k-2)}{432}$.

25. Let the joint probability density function of continuous random variables x_1 and x_2 be given by

$$f(x_1, x_2) = \begin{cases} 2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

then find the $E(x_1 | x_2)$, $\text{Var}(x_1 | x_2)$

- (A) $\frac{x_2}{2}$, $0 < x_2 < 1$, $\frac{x_2^2}{12}$, $0 < x_2 < 1$.
- (B) $\frac{x_2}{3}$, $0 < x_2 < 1$, $\frac{x_2^2}{2}$, $0 < x_2 < 1$.
- (C) $\frac{x_2}{12}$, $0 < x_2 < 1$, $\frac{x_2^2}{42}$, $0 < x_2 < 1$.
- (D) None of these
26. Solution of the differential equation

$$(2x \cos y + y^2 \cos x)dx + (2y \sin x - x^2 \sin y)dy = 0 \text{ is}$$

- (A) $x^2 \cos y + y^2 \sin x = c$
- (B) $x^2 \cos y + z^2 \sin x = c$
- (C) $y^2 \cos y + x^2 \sin x = c$
- (D) None of these

27. Three six-faced dice are thrown together. The probability that the sum of the numbers operating on the dice is

$k(3 \leq k \leq 8)$, is

(A) $\frac{k^2}{432}$

(B) $\frac{k(k-1)}{432}$

(C) $\frac{(k-1)(k-2)}{432}$

(D) $\frac{k(k-1)(k-2)}{432}$

28. If X is a $G\left(5, \frac{1}{3}\right)$ random variables, then $P(X > 6)$ equals

(A) e^{-1}

(B) e^{-2}

(C) $\frac{e^{-1}}{3}$

(D) $\frac{e^{-2}}{3}$

29. Let f be four times differentiable function on $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ such that $f'(x) = 1 + [f(x)]^3$. If $f(0) = 1$,

then the coefficient of x^3 in Taylor's expansion of f about zero is

(A) 6

(B) 7

(C) 8

(D) 9

30. Find the value of $\lim_{x \rightarrow \infty} \frac{\int_0^{2x} te^{t^2} dt}{e^{4x^2}}$.

(A) ∞

(B) 2

(C) $\frac{1}{2}$

(D) 0

SECTION - B (31-40) : MULTIPLE SELECT QUESTIONS (MSQs)

31. Consider the function $f(x,y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & \text{otherwise} \end{cases}$, then which of the following is/are

false for the given function ?

(A) $f_{xy}(0, 0) = f_{yx}(0, 0) = 1$

(B) $f_{xy}(0, 0) = f_{yx}(0, 0) = -1$

(C) $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

(D) $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$

32. What is/are solution of the differential equation $(a^2 - 2xy - y^2)dx - (x + y)^2dy = 0$, $y(0) = 1$?

(A) $a^2x - x^2y - y^2x - \frac{y^3}{3} = -\frac{1}{3}$

(B) $a^2x - xy^2 - x^2y + \frac{y^3}{3} = +\frac{1}{3}$

(C) $3a^2x - 3x^2y - 3y^2x - y^3 + 1 = 0$

(D) $a^2x - xy(x + y) + \frac{xy}{3} = 0$

33. If X_1 and X_2 are i.i.d. $N(0, 1)$ random variables, then $P(X_1^2 + X_2^2 \leq 1)$

(A) $\frac{e-1}{e}$

(B) $\frac{e-\sqrt{e}}{e}$

(C) $\frac{\sqrt{e}-1}{\sqrt{e}}$

(D) $\frac{e^2-1}{e^2}$

34. Let X_1, X_2, \dots, X_{10} be a random sample from a $N(\mu, 1)$ distribution. For testing $H_0 : \mu = 8$ against $H_1 : \mu = 9$ the most powerful critical region is $\bar{X} \geq K$, then which of the following value of k is/are incorrect for the size of the test is 0.05 ?

(A) $K = 14.5$

(B) $K = 8.52$

(C) $K = 17.04$

(D) $K = 8.00$

35. Let X be a random variable with the probability density function

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

then which of the following is/are not most powerful critical region (rejection region) for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ with size of critical region is α ?

(A) $\frac{x_1 + 2x_2 + \dots + nx_n}{n} \geq k_1$

(B) $\frac{x_1 + x_2 + \dots + x_n}{n} \geq k_1$

(C) $(x_1, x_2, \dots, x_n)^{1/n} \leq k_1$

(D) $(x_1 x_2 \dots x_n)^{1/n} \geq k_1$

36. If \bar{E} and \bar{F} are the complementary events of events E and F respectively and if $0 < P(F) < 1$, then

(A) $P\left(\frac{E}{F}\right) + P\left(\frac{\bar{E}}{F}\right) = 1$

(B) $P\left(\frac{E}{F}\right) + P\left(\frac{E}{\bar{F}}\right) = 1$

(C) $P\left(\frac{\bar{E}}{F}\right) + P\left(\frac{E}{\bar{F}}\right) = 1$

(D) $1 - P\left(\frac{\bar{E}}{F}\right) - P\left(\frac{E}{\bar{F}}\right) = 0$

37. A box contains N coins, m of which are fair and the rest are biased. The probability of getting a head when a fair coin is tossed is $\frac{1}{2}$, while it is $\frac{2}{3}$ when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. Then the probability that the coin drawn is fair, is

(A) $\frac{9m}{8N+m}$

(B) $\frac{18m^2}{16Nm+2m^2}$

(C) $\frac{9m}{8m-N}$

(D) $\frac{9m}{8m+N}$

38. If the moment generating function of a random variable X is given as $\frac{1}{(1-t)^5}, |t| < 1$ then

random variable X does not follow ---

(A) Exponential distribution with parameter 5

(B) χ^2 -distribution with degree of freedom 5

(C) gamma distribution with parameter 5

(D) Poisson distribution with parameter 5.

39. Which of the following is/are not equal to the value of integral $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2} dx dy$?

(A) $\frac{\pi a^2}{30}$

(B) $\frac{\pi a^2}{20}$

(C) $\frac{\pi a^2}{20}$

(D) $\frac{\pi a^2}{30}$

40. Find the general solution of the differential equation $(1 - x^2) \frac{dy}{dx} + xy = xy^2(1 - x^2)$, $y(0) = 1$

(A) $y\sqrt{x^2 - 1} = \left[2 - \frac{1}{3}(x^2 - 1)^{3/2}\right]$

(B) $\frac{3}{2}\sqrt{1 - x^2} = y \left[1 + \frac{(1 - x^2)^{3/2}}{2}\right]$

(C) $\sqrt{-x^2 + 1} = \frac{y}{3} \left[2 + (-x^2 + 1)^{3/2}\right]$

(D) $\sqrt{1 - x^2} = y \left[\frac{2}{3} + \frac{1}{3}(1 - x^2)^{3/2}\right]$

SECTION - C (41-60) : NUMERICAL ANSWER TYPE (NATs)

41. The following observations constitute a random sample from an unknown population. Find the estimator of the standard deviation of the population.

14, 19, 17, 20, 25

42. Find the area bounded by the parabolas $y = 4x^2$, $y = x^2/9$ and the straight line $y = 2$.

43. If A is a 7×9 matrix with a two dimensional null space, then what is the Rank of A ?

44. If $X \sim N(\mu, 9^2)$ and $Y \sim N(\mu, 12^2)$ are independent and if $P(X + 2Y \leq 3) = P(2X - Y \geq 4)$, determine μ .

45. Let $f(x_1, x_2) = \begin{cases} 6x_1^2x_2 & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere,} \end{cases}$ be the pdf of two random variables X_1 and X_2 of

the continuous type then find $P\left(0 < x_1 < \frac{3}{4}, \frac{1}{3} < x_2 < 2\right)$.

46. Let X_1 and X_2 have the joint pdf

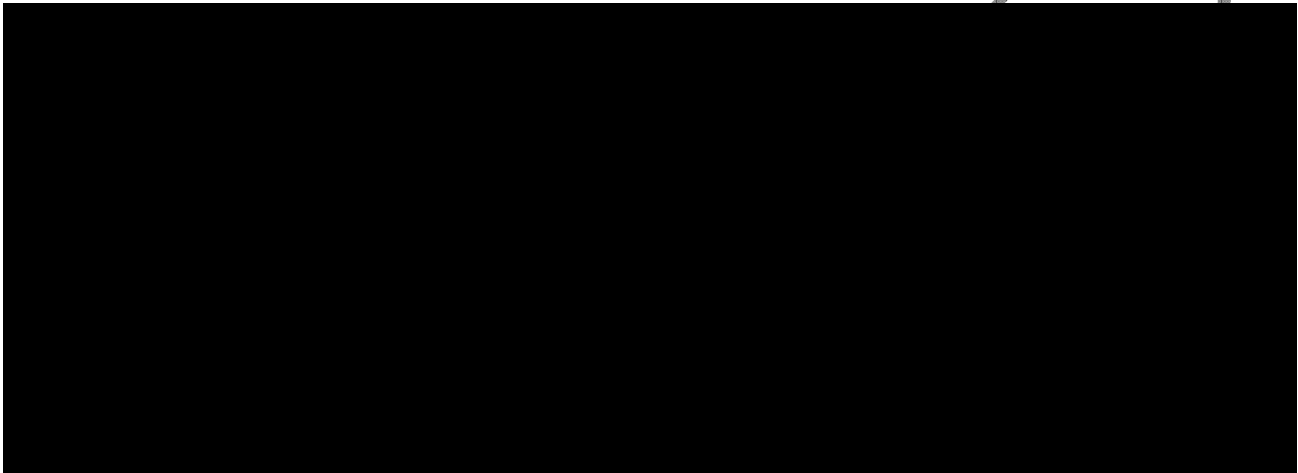
$$f(x_1, x_2) = \begin{cases} 6x_2, & 0 < x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}$$

then find $E\left(x_2 \mid x_1 = \frac{1}{2}\right)$.

47. Consider the sequence $\{x_n\}$, where $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$, then what is the value of ξ ?
48. Suppose that a box contains 3 white balls and 2 black balls. Let E_1 be the event "first ball drawn is black" and E_2 the event "second ball drawn is black," where the balls are not replaced after being drawn. Here E_1 and E_2 are dependent events, then what will be the probability if both balls drawn are black?
49. One hundred identical coins, each with probability p , of showing up a head, are tossed. If $0 < p < 1$ and if the probability of heads on exactly 50 coins is equal to that of heads on exactly 51 coins then find the value of p ?
50. What is the mean of the binomial distribution $\binom{10}{x} \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{10-x}$; $x = 0, 1, 2, \dots, 10$?
51. If X and Y are independent Poisson variates with means 1 and 3 respectively then find the value of variance of $3X + Y$?
52. A man is known to speak truth in 75% cases. If he throws an unbiased die and tells his friend that it is a six, then find the probability that it is actually a six.

53. A wire of length ℓ is cut into three pieces. What is the probability that the three pieces form a triangle?
54. Let the function defined by $f(x) = \min \{|x|, |x - 1|, |x + 1|\}$ then find the area (in sq. unit), which is bounded by $f(x)$ in $[-1, 1]$ and the x-axis.
55. If the region of the integration is $x + y \leq 1, x \geq 0, y \geq 0$, then what will be the value of $\iint xy \, dx \, dy$?
56. If y_1 and y_2 are the two linearly independent solution of differential equation $\frac{d^2y}{dx^2} = 9y = 0$ with $w(1) = +3$ then find $w(s)$.
57. What will be the value of limit $\lim_{x \rightarrow \infty} \frac{2x^{1/2} + 3x^{1/3} + 4x^{1/4} + \dots + nx^{1/n}}{(2x-3)^{1/2} + (2x-3)^{1/3} + \dots + (2x-3)^{1/n}}$?
58. Let f be a thrice differentiable function on $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ such that $f'(x) = 1 + [f(x)]^{-1}$. If $f(0) = -1$ then what will be the coefficient of x^2 in Taylor's expansion of f about zero ?
59. For which value of q the minimax estimator $\frac{X + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}}$ is unbiased estimator of binomial parameter ?
60. If the mgf of a r.v. x is $e^{4(e^x - 1)}$ then find the value of $P(\mu - 2\sigma < x < \mu + 2\sigma)$.

ANSWER KEY



HINTS AND SOLUTION

1.(B) Here $P(H) = P(T) = \frac{1}{2}$ and $P(X) = 1$, where X denotes head or tail.

If the sequence of m consecutive heads starts from the first throw, we have (HH.... m times) (XX n times).

$$\therefore \text{Chance of this event} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \dots m \text{ times} = \frac{1}{2^m}$$

$m + 1$ and subsequent throws may be head or tail since we are considering at least m consecutive heads. If the sequence of m consecutive heads starts from the second

thrown, the first must be a tail and we have, the chance of this event $= \frac{1}{2} \cdot \frac{1}{2^m} = \frac{1}{2^{m+1}}$.

If the sequence of heads starts from $(r + 1)^{\text{th}}$ throw then the first $(r - 1)$ throws may be head or tail but r^{th} throw must be a tail and we have,

(XX..... $(r - 1)$ times) T(HH....., n times)

(XX..... $n - m - r$ times)

The chance of this event also $\frac{1}{2} \times \frac{1}{2^m} = \frac{1}{2^{m+1}}$

Since all the above events are mutually exclusive, so the required probability

$$= \frac{1}{2^m} + \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} + \dots n \text{ times} \right) = \frac{1}{2^m} + \frac{n}{2^{m+1}} = \frac{n+2}{2^{m+1}}$$

2.(A) The required mean is,

$$\begin{aligned} \bar{x} &= \frac{0 \cdot q^n + 1 \cdot \frac{n}{1} q^{n-1} p + 2 \cdot \frac{(n)(n-1)}{2!} q^{n-2} p^2 + \dots + n \cdot p^n}{q^n + \frac{n}{1} q^{n-1} p + \frac{n(n-1)}{2} q^{n-2} p^2 + \dots + p^n} \\ &= \frac{0 \cdot {}^n C_0 q^n p^0 + 1 \cdot {}^n C_1 q^{n-1} p + \dots + n \cdot {}^n C_n q^0 p^n}{{}^n C_0 q^n p^0 + {}^n C_1 q^{n-1} p^1 + \dots + {}^n C_n q^0 p^n} \\ &= \frac{\sum_{r=0}^n r \cdot {}^n C_r q^{n-r} p^r}{\sum_{r=0}^n {}^n C_r q^{n-r} p^r} = \frac{\sum_{r=1}^n r \cdot \frac{n-1}{r} {}^{n-1} C_{r-1} q^{n-r} \cdot p \cdot p^{r-1}}{\sum_{r=0}^n {}^n C_r q^{n-r} p^r} \\ &= \frac{np \left(\sum_{r=1}^n (n-1) {}^{n-1} C_{r-1} p^{r-1} q^{(n-1)-(r-1)} \right)}{\sum_{r=0}^n {}^n C_r q^{n-r} p^r} \end{aligned}$$

$$= \frac{np(q+p)^{n-1}}{(q+p)^n} = np, [\because q+p=1]$$

3.(D)

$$f(x) = \sin^3 x + \lambda \sin^2 x$$

$$f'(x) = 3 \sin^2 x \cos x + 2\lambda \sin x \cos x = \frac{1}{2} \sin 2x (3 \sin x + 2\lambda) \dots (1)$$

For maximum or minimum values of $f(x)$, $f'(x) = 0$

$$\frac{1}{2} \sin 2x (3 \sin x + 2\lambda) = 0$$

$$\sin 2x = 0$$

$$\Rightarrow 2x = n\pi$$

$$\Rightarrow x = \frac{n\pi}{2}$$

$$\therefore x = 0 \left(\because -\frac{\pi}{2} < x < \frac{\pi}{2} \right)$$

$$\text{and } 3 \sin x + 2\lambda = 0$$

$$\therefore \sin x = -\frac{2\lambda}{3}$$

.....(2)

$$\text{or } -1 < \sin x < 1$$

$$\Rightarrow -1 < -\frac{2\lambda}{3} < 1$$

$$\Rightarrow -\frac{3}{2} < \lambda < \frac{3}{2}$$

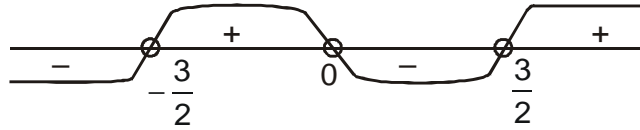
.....(3)

Also from (1),

$$f''(x) = \cos 2x (3 \sin x + 2\lambda) \frac{3}{2} + \sin 2x \cos x$$

$$f'' \left(\sin^{-1} \left(-\frac{2\lambda}{3} \right) \right) = 0 + 3 \left(1 - \frac{4\lambda^2}{9} \right) \left(-\frac{2\lambda}{3} \right)$$

$$= \frac{-2\lambda(9-4\lambda^2)}{9} = \frac{2}{9}\lambda(2\lambda+3)(2\lambda-3)$$



If $f''\left(\sin^{-1}\left(\frac{-2\lambda}{3}\right)\right) > 0$ (f(x) is minimum)

then $\lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(\frac{3}{2}, \infty\right)$ (4)

and If $f''\left(\sin^{-1}\left(-\frac{2\lambda}{3}\right)\right) < 0$ (f(x) is maximum)

then $\lambda \in \left(-\infty, -\frac{3}{2}\right) \cup \left(0, \frac{3}{2}\right)$ (5)

But from (3), $-\frac{3}{2} < \lambda < \frac{3}{2}$ (6)

Combining (4), (5), and (6), we get

$$\lambda \in \left(-\frac{3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$$

4.(B) In the given determinant. Replacing R_1 by $R_1 + R_2 + R_3$

$$\begin{vmatrix} a+b+c-x & a+b+c-x & a+b+c-x \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

Taking out $(a + b + c - x)$ as common

$$\Rightarrow (a + b + c - x) \begin{vmatrix} 1 & 1 & 1 \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

Replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively

$$\Rightarrow (a + b + c - x) \begin{vmatrix} 1 & 0 & 0 \\ c & b-c-x & a-c \\ b & a-b & c-b-x \end{vmatrix} = 0$$

Expanding with respect to first row.

$$\Rightarrow (a + b + c - x) \begin{vmatrix} b-c-x & a-c \\ a-b & c-b-x \end{vmatrix} = 0,$$

$$\Rightarrow (a + b + c - x)[(b - c - x)(c - b - x) - (a - c)(a - b)] = 0$$

$$\Rightarrow a + b + c - x = 0$$

$$\Rightarrow x = a + b + c$$

$$\Rightarrow x = 0 \text{ (Given that } a + b + c = 0)$$

5.(C) Here, given equations are

$$x - y + 3z = 4$$

$$x + z = 2$$

$$x + y - z = 0$$

Write the system of equations in matrix form $AX = B$.

We get

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{Coefficient matrix } A = \begin{bmatrix} 1 & -1 & 3:4 \\ 1 & 0 & 1:2 \\ 1 & 1 & -1:0 \end{bmatrix}$$

$$\text{Rank of } A = 2 = \begin{bmatrix} 1 & -1 & 3:4 \\ 0 & 1 & -2:-2 \\ 0 & 1 & -2:-2 \end{bmatrix}$$

$$\text{By } R_2 = R_2 - R_1$$

$$R_3 = R_3 - R_2 = \begin{bmatrix} 1 & -1 & 3:4 \\ 0 & 1 & -2:-2 \\ 0 & 1 & 0:-2 \end{bmatrix}$$

$$\text{By } R_3 = R_3 - R_2$$

$$\text{Rank } [A : B] = 2$$

$$\text{Rank } (A) = \text{Rank } [A : B] < \text{Number of unknowns}$$

Therefore, the given system of equation has infinite many solutions.

- 6.(D)** Let p denote the probability of getting a head in a single toss and $q = 1 - p$ be the probability of getting a tail. Consider the variable $x =$ Number of tosses required to obtain the first head. x can take the values $1, 2, 3, \dots, n$. In general, r tosses will be required if the first $r - 1$ tosses result in tails only and the last toss gives a head. Since the successive trials are independent the probability of such an occurrence, i.e.

$$P(x = r) = (q \cdot q \dots q) \cdot p = p \cdot q^{r-1}$$

$$\underbrace{(q \cdot q \dots q)}_{r-1 \text{ times}} \cdot p = p \cdot q^{r-1}$$

$$\text{Hence } E(x) = \sum_{r=1}^{\infty} (p \cdot q^{r-1}) \cdot r = p \sum_{r=1}^{\infty} (r \cdot q^{r-1}) = p(1 + 2q + 3q^2 + \dots) = p \cdot (1 - q)^{-2}$$

$$\text{Since } (1 - x)^{-2} = 1 + 2x + 3x^2 + \dots \text{ for } |x| < 1 = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

7.(C) We have

$$E\left(\frac{X_1 + X_2 + \dots + X_n}{X_1 + X_2 + \dots + X_n}\right) = E(1) \Rightarrow E\left(\frac{X_1 + X_2 + \dots + X_n}{S_n}\right) = 1$$

$$E\left(\frac{X_1}{S_n}\right) + E\left(\frac{X_2}{S_n}\right) + \dots + E\left(\frac{X_n}{S_n}\right) = 1$$

Since X_i 's, ($i = 1, 2, \dots, n$) are identically distributed random variables, (X_i/S_n) , ($i = 1, 2, \dots, n$) are also identically distributed random variables.

$$n E\left(\frac{X_i}{S_n}\right) = 1 \Rightarrow E\left(\frac{X_i}{S_n}\right) = \frac{1}{n}; i = 1, 2, \dots, n \quad \dots(*)$$

Now,

$$\begin{aligned} E\left(\frac{S_m}{S_n}\right) &= E\left(\frac{X_1 + X_2 + \dots + X_m}{S_n}\right) = E\left(\frac{X_1}{S_n} + \frac{X_2}{S_n} + \dots + \frac{X_m}{S_n}\right) \\ &= E\left(\frac{X_1}{S_n}\right) + E\left(\frac{X_2}{S_n}\right) + \dots + E\left(\frac{X_m}{S_n}\right) = \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} \quad [(m \text{ times})][\text{Using}(*)] = \frac{m}{n}, (m < n) \end{aligned}$$

8.(D) Let X be a r.u. with probability function :

$$P(X = \pm 2^x) = \begin{cases} \frac{e^{-1}}{2^x!}; & x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

Since the distribution is symmetric about the line $X = 0$, all moments of odd order about origin vanish, i.e.

$$E(X^{2r+1}) = 0 \quad \mu_{2r+1} = 0$$

$$E(X^{2r}) = \sum_{x=0}^{\infty} (\pm 2^x) \left(\frac{1}{2e^x} \right)^{2r} = \frac{1}{e} \sum_{x=0}^{\infty} \frac{(2^{2r})^x}{x!} = \frac{1}{e} \exp(2^{2r}) = \exp(2^{2r} - 1)$$

Thus all the moments of X exist. The m.g.f of X , if it exists, is given by

$$M_X(t) = \sum_{x=0}^{\infty} \left\{ (e^{t \cdot 2^x} + e^{-t \cdot 2^x}) \frac{1}{2e^{2^x}} \right\} = e^{-1} \sum_{x=0}^{\infty} \left\{ \frac{\cosh(t2^x)}{e^{2^x}} \right\}$$

Which converges only for $t = 0$.

As an illustration of continuous probability distribution consider Pareto distribution with p.d.f:

$$p(x) = \frac{\theta a^\theta}{x^{\theta+1}}; x \geq a, \theta > 1$$

$$E(X^r) = \theta a^\theta \int_a^\infty x^{r-\theta-1} dx = \theta a^\theta \left[\frac{x^{r-\theta}}{r-\theta} \right]_a^\infty, \text{ which is finite iff } r - \theta < 0 \Rightarrow \theta > r$$

and then
$$E(X^r) = \theta a^\theta \left[0 - \frac{a^{r-\theta}}{r-\theta} \right] = \frac{\theta a^r}{\theta - r}; \theta > r$$

However, the m.g.f. is given by :

$$M_X(t) = \theta a^\theta \int_a^\infty \frac{e^{tx}}{x^{\theta+1}} dx,$$

Which does not exist, since e^{tx} dominates $x^{\theta+1}$ and $(e^{tx} / x^{\theta+1}) \rightarrow \infty$ as $x \rightarrow \infty$ and hence the integral is not convergent.

9.(B) We shall assume that X and Y are continuous random variables.

$$P(AB) = P[(x : a < x < b)(y : c < y < d)] = \int_a^b \int_c^d f_{xy}(x, y) dx dy = \int_a^b \int_c^d g_x(x) h_y(y) dx dy,$$

$$\text{since } A \text{ and } B \text{ are independent} = \left[\int_c^d g_x(x) dx \right] \left[\int_c^d h_y(y) dy \right] = P(A)P(B)$$

which proves the result.

10.(B) Marginal P.d.f of X is given by :

$$f_x(x) = \int_0^1 f(x, y) dy = \int_0^1 (x+y) dy = \left[xy + \frac{y^2}{2} \right]_0^1 = \left(\frac{x+1}{2} \right), 0 \leq x \leq 1$$

then the condition distribution of y for $X = x$ is given by :

$$f_{xy}(Y = y | X = x) = \frac{f_{xy}(x, y)}{f_x(x)} = \frac{x+y}{\frac{1}{2}(2x+1)} = \frac{2(x+y)}{2x+1}$$

$$f_{xy}(Y = y | X = x) = \frac{2(x+y)}{2x+1}; 0 \leq y \leq 1.$$

11.(A) L.H.L = $\lim_{x \rightarrow \pi/4^-} f(x) = \lim_{h \rightarrow 0} a \left(\frac{\pi}{4} - h \right) + b = a \left(\frac{\pi}{4} \right) + b$

R.H.L = $\lim_{x \rightarrow \pi/4^+} f(x) = \lim_{h \rightarrow 0} \cos \left(\frac{\pi}{4} + h \right) = \frac{1}{\sqrt{2}}$

For continuity L.H.L = R.H.L

$$\Rightarrow a\left(\frac{\pi}{4}\right) + b = \frac{1}{\sqrt{2}} \quad \dots(1)$$

$$\text{L.f.}\left(\frac{\pi}{4}\right) = \lim_{h \rightarrow 0} \frac{f\left(\frac{\pi}{4}-h\right) - f\left(\frac{\pi}{4}\right)}{-h} = \lim_{h \rightarrow 0} \frac{\left\{a\left(\frac{\pi}{4}-h\right) + b\right\} - \left\{a\left(\frac{\pi}{4}\right) + b\right\}}{-h} = a$$

$$\begin{aligned} \text{R.f.}\left(\frac{\pi}{4}\right) &= \lim_{h \rightarrow 0} \frac{f\left(\frac{\pi}{4}+h\right) - f\left(\frac{\pi}{4}\right)}{h} = \lim_{h \rightarrow 0} \frac{\cos\left(\frac{\pi}{4}+h\right) - \cos\frac{\pi}{4}}{h} = \lim_{h \rightarrow 0} \frac{-2\sin\left(\frac{\pi}{4} + \frac{h}{2}\right) \cdot \sinh}{h} \\ &= -2\sin\frac{\pi}{4} \cdot \lim_{h \rightarrow 0} \frac{\sinh}{h} = -2 \times \frac{1}{\sqrt{2}} \times 1 = \frac{-1}{\sqrt{2}} \end{aligned}$$

$$\therefore \text{L.f.}\left(\frac{\pi}{4}\right) = \text{R.f.}\left(\frac{\pi}{4}\right)$$

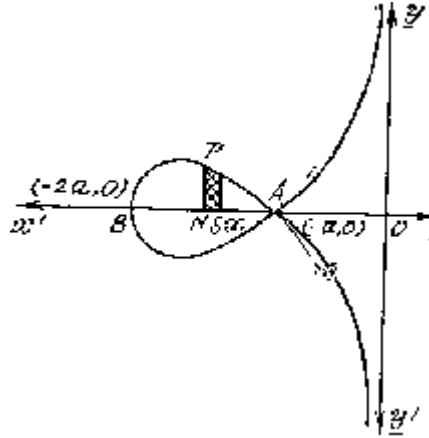
$$\Rightarrow a = -\frac{1}{\sqrt{2}}$$

Hence, from (1)

$$b = \frac{1}{\sqrt{2}} \left(\frac{\pi}{4} + 1\right)$$

12.(D) The curve $x^2 + y^2 = a^2$ is a circle with centre 0 (0, 0) and radius a.

The equation $x + y = a$ or $(x/a) + (y/a) = 1$ represents a straight line which cuts off intercepts a and a from positive directions of x and y axis.



Hence the points of intersection of $x^2 + y^2 = a^2$ and $x + y = a$ are A (a, 0) and B (0, a) respectively.

So the area under consideration is the area AQPBP'Q'A

\therefore The required area = $\int_{x=0}^a \int_{y=P'N}^{PN} dx dy$, where $PN = \sqrt{(a^2 - x^2)}$ and $P'N = a - x$

$$= \int_{x=0}^a \int_{y=(a-x)}^{\sqrt{(a^2-x^2)}} dx dy = \int_0^a [y]_{(a-x)}^{\sqrt{(a^2-x^2)}} dx = \int_0^a [\sqrt{(a^2-x^2)} - (a-x)] dx$$

$$= \left[\left\{ \frac{1}{2} x \sqrt{(a^2-x^2)} + \frac{1}{2} a^2 \sin^{-1}(x/a) \right\} - ax + \frac{1}{2} x^2 \right]_0^a$$

$$= \left[\frac{1}{2} a^2 \left(\frac{1}{2} \pi \right) - a^2 + \frac{1}{2} a^2 \right] = \frac{1}{2} a^2 \left(\frac{1}{2} \pi - 1 \right) = \frac{1}{4} a^2 (\pi - 2).$$

13.(D) Here first of all we shall determine the constant y_0 from the consideration that the area under a probability curve is unity.

$$\therefore y_0 \int_{\alpha}^{\infty} \exp[-\beta(x-\alpha)] dx \Rightarrow y_0 \left[\frac{e^{-\beta(x-\alpha)}}{-\beta} \right]_{\alpha}^{\infty} = 1 \Rightarrow -\frac{y_0}{\beta} (0-1) = 1 \Rightarrow y_0 = \beta$$

$$\therefore f(x; \alpha, \beta) = \beta e^{-\beta(x-\alpha)}, \alpha \leq x < \infty$$

If $x_1, x_2, x_3, \dots, x_n$ is a random sample of n observation from this population, then

$$L = \prod_{i=1}^n f(x_i; \alpha, \beta) = \beta^n \exp \left\{ -\beta \sum_{i=1}^n (x_i - \alpha) \right\} = \beta^n \exp [-n\beta(\bar{x} - \alpha)]$$

$$\therefore \log L = n \log \beta - n\beta (\bar{x} - \alpha) \quad \dots\dots(i)$$

The likelihood equations for estimating α and β give

$$\frac{\partial}{\partial \alpha} \log L = 0 = n\beta \quad \dots\dots(ii)$$

$$\text{and } \frac{\partial}{\partial \beta} \log L = 0 = \frac{n}{\beta} - n(\bar{x} - \alpha) \quad \dots\dots(iii)$$

Equation (2) given $\beta = 0$, which is obviously inadmissible and this on substitution in (3) gives $\alpha = \infty$. Thus the likelihood equations fail to give us valid estimates of α and β and we try to locate M.L.Es for α and β by maximising L directly.

L is maximum \Rightarrow $\log L$ maximum

From (1), $\log L$ is maximum (for any value of β), if $(\bar{x} - \alpha)$ is minimum, which is so if α is maximum and

If $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ is ordered sample from this population then $\alpha \leq x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$ so that the maximum value of α consistent with the sample is $x_{(1)}$, the smallest sample observation, i.e., $\hat{\alpha} = x_{(1)}$.

Consequently, (3) gives $\frac{1}{\beta} = \bar{x} - \hat{\alpha} = \bar{x} - x_{(1)} \Rightarrow \hat{\beta} = \frac{1}{\bar{x} - x_{(1)}}$

Hence M.L.E's for α and β are given by : $\hat{\alpha} = x_{(1)}$ and $\hat{\beta} = \frac{1}{\bar{x} - x_{(1)}}$.

14.(A) $f(x, \theta) = \frac{x}{1+\theta}$; $0 \leq x \leq 1$

$$\mu_1 = E(x) = \int_0^1 x \cdot \frac{x}{1+\theta} dx = \frac{1}{1+\theta} \int_0^1 x^2 dx = \frac{1}{1+\theta} \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{1+\theta} \left[\frac{1}{3} \right] = \frac{1}{3(1+\theta)}$$

$$\Rightarrow 1 + \theta = \frac{1}{3\mu_1} \Rightarrow \theta = \frac{1}{3\mu_1} - 1$$

Replace μ_1 by M_1 .

$$\therefore \theta = \frac{1}{3M_1} - 1.$$

15.(A) $f(x) = x^2 e^{-x}$

$$f'(x) = 2x \cdot e^{-x} - x^2 e^{-x}$$

$$f'(x) = 0 \quad (\text{for maxima and minima})$$

$$x e^{-x} (2 - x) = 0$$

$$x = 0, x = 2$$

$$f''(x) = 2e^{-x} - 2xe^{-x} - 2x \cdot e^{-x} + x^2 e^{-x} = e^{-x} [2 - 2x - 2x + x^2] = e^{-x} [x^2 - 4x + 2]$$

$$f''(0) > 0 \quad x = 0 \text{ point of minima}$$

$$f''(2) < 0 \quad x = 2 \text{ point of maxima}$$

$$f(0) = 0$$

$$f(2) = 4e^{-2} = \frac{4}{e^2}$$

16.(A) The given equation can be rewritten as

$$-(x+a) \frac{dy}{dx} = ay^2 \quad \text{or} \quad \frac{(x+a)}{y^2} \frac{dy}{dx} + \frac{1}{y} = 0 \dots(1)$$

Putting $\frac{1}{y} = v$ or $-\frac{1}{y^2} \frac{dy}{dx} = \frac{dv}{dx}$ in (1) we get

$$(x+a) \frac{dv}{dx} + v = a \quad \text{or} \quad \frac{dv}{dx} + \frac{v}{x+a} = \frac{a}{x+a} \dots(2)$$

which is a linear equation in v.

Its integrating factor = $e^{\int [1/(x+a)] dx} = e^{\log(x+a)} = x+a$

Multiplying both sides of (2) by this integrating factor and integrating

we have $v(x+a) = C + \int \frac{a}{x+a} \cdot (x+a) dx = C + a \int dx$,

where C is an arbitrary constant.

or $v(x+a) = C + ax$ or $(x+a)/y = C + ax$, $v = 1/y$

or $x+a = y(C+ax)$

Put $x=0, y=1$ we get $\boxed{c=a}$

$$\frac{x+a}{a} = y(1+x)$$

17.(B) The C-coordinate vectors of the image of \mathbf{b}_1 and \mathbf{b}_2 are

$$[T(\mathbf{b}_1)]_C = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{b}_2)]_C = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

Hence

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

If B and C are bases for the same space V and if T is the identity transformation $T(x) = x$ for x in V, then the matrix M in (4) is just a change-of-coordinates matrix.

18.(C) $X \sim N(0, 1)$, $\therefore E(x) = 0, \text{Var } x = 1, Ex^2 = 1, Ex^3 = 0, Ex^4 = 3.$

$$E(Y) = E(a + bx + cx^2) = a + b \cdot 0 + c \cdot 1 = a + b \cdot 0 + c \cdot 1 = a + c, \sigma_y^2 = E[Y - (a + c)]^2$$

$$= E(Y^2) - (a + c)^2 = b^2 + 2c^2 - ac$$

$$\therefore r(X, Y) = \text{Cov}(X, Y) / \sigma_x \sigma_y = \frac{b}{\sqrt{b^2 + 2c^2 - ac}}$$

19.(A) Given that X_1 and X_2 have the joint pdf

$$f_{x_1, x_2}(x_1, x_2) = \begin{cases} 10x_1x_2^2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

where $Y_1 = X_1 / X_2$ and $Y_2 = X_2$. Hence, the inverse transformation is $x_1 = y_1y_2$ and $x_2 = y_2$ which has the Jacobian

$$J = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2.$$

The inequalities defining the support S of (X_1, X_2) becomes

$$0 < y_1y_2, y_1y_2 < y_2, \text{ and } y_2 < 1.$$

These inequalities are equivalent to

$$0 < y_1y_2, y_1y_2 < y_2, \text{ and } y_2 < 1.$$

These inequalities are equivalent to

$$0 < y_1 < 1 \text{ and } 0 < y_2 < 1,$$

which defines the support set T of (Y_1, Y_2) . Hence, the pdf of (Y_1, Y_2) is

$$f_{Y_1, Y_2}(y_1, y_2) = 10y_1 y_2 y_2^2 | y_2 | = 10y_1 y_2^4, \quad (y_1, y_2) \in T.$$

The marginal pdf are :

$$f_{Y_1}(y_1) = \int_0^1 10y_1 y_2^4 dy_2 = 2y_1, \quad 0 < y_1 < 1$$

zero elsewhere.

20.(C) Let X_1 and X_2 have the joint pdf.

$$f(x_1, x_2) = \begin{cases} 6x_2 & 0 < x_2 < x_1 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then the marginal pdf of X_1 is

$$f_1(x_1) = \int_0^{x_1} 6x_2 dx_2 = 3x_1^2, \quad 0 < x_1 < 1,$$

zero elsewhere. The conditional pdf of X_2 , given $X_1 = x_1$, is

$$f_{2|1}(x_2 | x_1) = \frac{6x_2}{3x_1^2} = \frac{2x_2}{x_1^2}, \quad 0, x_2 < x_1,$$

zero elsewhere, where $0 < x_1 < 1$. The conditional mean of X_2 , given $X_1 = 1/2$, is

$$E\left(X_2 | x_1 = \frac{1}{2}\right) = \int_0^{x_1} x_2 \left(\frac{2x_2}{x_1^2}\right) dx_2 = \frac{2}{3} x_1, \quad 0 < x_1 < 1.$$

$$\text{i.e. } \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

21.(A) The pmf of X on its support is

$$f(x, \theta) = e^{-\theta} \frac{\theta^x}{x!} = \exp\{\log(\theta)x + \log(1/x!) + (-\theta)\}.$$

Hence, the Poisson distribution is a member of the regular exponential class, with $p(\theta) = \log(\theta)$, $q(\theta) = -\theta$, and $K(x) = x$. Therefore if X_1, X_2, \dots, X_n denotes a random sample on X then the statistic $Y_1 = \sum_{i=1}^n X_i$ is sufficient.

22.(A) Null Hypothesis, (H_0) : The sample has been drawn from the population with mean $\mu = 3.25$ cms. and S.D. $\sigma = 2.61$ cms.

Alternative Hypothesis, $H_1 : \mu \neq 3.25$ (Tw-tailed).

Test Statistic. Under H_0 , the test statistic is : $Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$, (since n is large.)

Here, we are given : $\bar{x} = 3.4$ cms., $n = 900$ cms., $\mu = 3.25$ cms. and $\sigma = 2.61$ cms.

$$\therefore Z = \frac{3.40 - 3.25}{2.61/\sqrt{900}} = \frac{0.15 \times 30}{2.61} = 1.73$$

Since $|Z| < 1.96$, we conclude that the data don't provide us any evidence against the null hypothesis (H_0) which may, therefore, be accepted at 5% level of significance.

95% fiducial limits for the population mean μ are :

$$\bar{x} \pm 1.96 \left(\frac{\sigma}{\sqrt{n}}\right) = 3.40 \pm 1.96 \left(\frac{2.61}{\sqrt{900}}\right) = 3.40 \pm 0.1705, \text{ i.e., } 3.5705 \text{ and } 3.2295$$

98% fiducial limits for μ are given by :

$$\bar{x} \pm 2.33 \frac{\sigma}{\sqrt{n}} = 3.40 \pm 2.33 \times \frac{2.61}{30} = 3.40 \pm 0.2027, \text{ i.e., } 3.6027 \text{ and } 3.1973$$

Remark. 2.33 is the value z_1 of Z from standard normal probability integrals, such that

$$P(|Z| > z_1) = 0.98 \Rightarrow P(Z > z_1) = 0.49$$

23.(A)
$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{\int_{\frac{\pi}{4}}^{\sec^2 x} f(t) dt}{x^2 - \frac{\pi^2}{16}} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{f(\sec^2 x) \cdot \frac{d}{dx}(\sec^2 x) - f(2) \cdot \frac{d}{dx}(2)}{2x} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{f(\sec^2 x) \cdot 2 \sec x \cdot \sec x \tan x - 0 \tan x}{2x}$$

$$= \frac{f(2) \cdot 2 \cdot 2 \cdot 1}{2 \cdot \frac{\pi}{4}} = \frac{8}{\pi} f(2) \quad \therefore \text{(A) holds.}$$

24.(A) The total number of elementary events associated to the random experiment of throwing three dice is $6 \times 6 \times 6 = 6^3$.

Favourable number of elementary events

$$= \text{Coefficient of } x^k \text{ in } (x + x^2 + x^3 + \dots + x^6)^3 = \text{Coefficient of } x^{k-3} \text{ in } \left(\frac{1-x^6}{1-x}\right)^3$$

$$= \text{Coefficient of } x^{k-3} \text{ in } (1-x^6)^3 (1-x)^{-3} = \text{Coefficient of } x^{k-3} \text{ in } (1-x)^{-3}$$

$$[\because 0 \leq k-3 \leq 5]$$

$$= k-3+3-1 C_{3-1} [\text{coefficient of } x^n \text{ in } (1-x)^{-r} = {}^{n+r-1}C_{r-1}] = {}^{k-1}C_2$$

$$= \frac{(k-1)(k-2)}{2}$$

$$\text{Hence, the probability of the required event} = \frac{(k-1)(k-2)}{2 \times 6^3} = \frac{(k-1)(k-2)}{432}.$$

25.(A) Given that X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} 2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then the marginal probability density functions are, respectively,

$$f_1(x_1) = \begin{cases} \int_{x_1}^1 2dx_2 = 2(1-x_1) & 0 < x_1 < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

and $f_2(x_2) = \begin{cases} \int_0^{x_2} 2dx_1 = 2x_2 & 0 < x_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$

The conditional pdf of X_1 , given $X_2 = x_2$, $0 < x_2 < 1$, is

$$f_{1|2}(x_1 | x_2) = \begin{cases} \frac{2}{2x_2} = \frac{1}{x_2} & 0 < x_1 < x_2 \\ 0 & \text{elsewhere.} \end{cases}$$

Here the conditional mean and the conditional variance of X_1 , given $X_2 = x_2$, are respectively,

$$E(X_1 | x_2) = \int_{-\infty}^{\infty} x_1 f_{1|2}(x_1 | x_2) dx_1 = \int_0^{x_2} x_1 f_{1|2}(x_1 | x_2) dx_1 = \frac{x_2}{2}, \quad 0 < x_2 < 1,$$

and $\text{var}(X_1 | x_2) = \int_0^{x_2} \left(x_1 - \frac{x_2}{2}\right)^2 \left(\frac{1}{x_2}\right) dx_1 = \frac{x_2^2}{12}, \quad 0 < x_2 < 1.$

26.(A) $(2x \cos y + y^2 \cos x) dx + (2y \sin x - x^2 \sin y) dy = 0$

$$\Rightarrow d\left(x^2 \cos y\right) + d\left(y^2 \sin x\right) = 0$$

On integrating, $x^2 \cos y + y^2 \sin x = c$

27.(C) The total number of elementary events associated to the random experiment of throwing three dice is $6 \times 6 \times 6 = 6^3$.

Favourable number of elementary events

$$= \text{Coefficient of } x^k \text{ in } (x + x^2 + x^3 + \dots + x^6)^3 = \text{Coefficient of } x^{k-3} \text{ in } \left(\frac{1-x^6}{1-x}\right)^3$$

$$= \text{Coefficient of } x^{k-3} \text{ in } (1-x^6)^3 (1-x)^{-3}$$

$$= \text{Coefficient of } x^{k-3} \text{ in } (1-x)^{-3} \quad [\because 0 \leq k-3 \leq 5]$$

$$= {}^{k-3+3-1}C_{3-1} [\text{coefficient of } x^n \text{ in } (1-x)^{-r} = {}^{n+r-1}C_{r-1}]$$

$$= {}^{k-1}C_2 = \frac{(k-1)(k-2)}{2}$$

$$\text{Hence, the probability of the required event} = \frac{(k-1)(k-2)}{2 \times 6^3} = \frac{(k-1)(k-2)}{432}$$

28.(D) Given that $X \sim G\left(5, \frac{1}{3}\right)$

$$\text{where } f(x | \alpha, \lambda) = \begin{cases} \frac{\lambda^\alpha}{\sqrt{\alpha}} x^{\alpha-1} e^{-\lambda x}, & x > 0, \lambda > 0, \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Now } P(X > 6) = \int_6^\infty \frac{1}{8(3)^6} x^4 e^{-x/3} dx = \frac{1}{8(3)^6} \int_0^\infty x^4 e^{-x/3} dx$$

$$= \frac{1}{8(3)^6} \left[\frac{x^4 e^{-x/3}}{-1/3} + \frac{4}{1/3} \int x^3 e^{-x/3} dx \right]_6^\infty = \frac{12}{8(3)^6} \left[0 + \frac{x^3 e^{-x/3}}{-1/3} + \frac{3}{1/3} \int x^2 e^{-x/3} dx \right]_6^\infty$$

$$= \frac{3^2 \cdot 3 \cdot 4}{8(3)^6} \left[0 + x^2 e^{-x/3} + \frac{2}{1/3} \int x e^{-x/3} dx \right]_6^\infty = \frac{3^4 \cdot 8}{8(3)^6} \left[0 + \frac{x e^{-x/3}}{-1/3} - 3^2 e^{-x/3} \right]_6^\infty = \frac{3^5 \cdot 8}{3^6 \cdot 8} [0 - e^{-\infty} + e^{-2}] = \frac{e^{-2}}{3}$$

29.(B) We know that Taylor's expansion of $f(x)$ about origin is

$$f(0+h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \frac{h^3}{3!}f'''(0) + \dots$$

$$\text{or } f(0+x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$\text{or } f(0+x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$\therefore f(0) = 1$$

$$f'(0) = 1 + [f(0)]^3 = 1 + [1]^3 = 2$$

$$f''(0) = 3[f(0)]^2 \times f'(0) = 3[1]^2 \times (2) = 6$$

$$f'''(0) = 3[f''(0) f(0)] + 6f(0) [f'(0)]^2 = 3[6(1)^2] + 6[1(2)^2] = 18 + 24 = 42$$

then coefficient of $f'''(0)$ in Taylor's expansion is given as

$$\frac{42}{3!} = \frac{42}{6} = 7$$

30.(C) Given $\lim_{x \rightarrow \infty} \frac{\int_0^{2x} te^{t^2} dt}{e^{4x^2}}$ ($\frac{\infty}{\infty}$ form)

$$\begin{aligned} \text{then } \lim_{x \rightarrow \infty} \frac{\int_0^{2x} te^{t^2} dt}{e^{4x^2}} &= \lim_{x \rightarrow \infty} \frac{\frac{d}{dx} \int_0^{2x} te^{t^2} dt}{\frac{d}{dx} e^{4x^2}} = \lim_{x \rightarrow \infty} \frac{\left[\int_0^{2x} \frac{\partial}{\partial x} te^{t^2} dt + 2xe^{(2x)^2} (2) - 0 \right]}{e^{4x^2} (8x)} = \lim_{x \rightarrow \infty} \frac{[0 + 4xe^{4x^2} - 0]}{8xe^{4x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{2} = \frac{1}{2} \end{aligned}$$

31.(A,B,D) Since $f_{xy}(0,0) = \lim_{h \rightarrow 0} \frac{f_y(h,0) - f_y(0,0)}{h}$

$$\text{But } f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \rightarrow 0} \frac{0}{k} = 0$$

$$f_y(h, 0) = \lim_{k \rightarrow 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k \rightarrow 0} \frac{hk(h^2 - k^2)}{k \cdot (h^2 + k^2)} = h$$

So, $f_{xy}(0, 0) = \lim_{h \rightarrow 0} \frac{h-0}{h} = 1$

again $f_{yx}(0, 0) = \lim_{k \rightarrow 0} \frac{f_x(0, k) - f_x(0, 0)}{k}$

but $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \frac{0}{h} = 0$

$$f_x(0, k) = \lim_{h \rightarrow 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h \rightarrow 0} \frac{hk(h^2 - k^2)}{h(h^2 + k^2)} = k$$

So, $f_{yx}(0, 0) = \lim_{k \rightarrow 0} \frac{-k-0}{+k} = -1$

Since $f_{xy}(0, 0) = 1$ and $f_{yx}(0, 0) = -1$

Hence $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

32.(A,C) Given differential equation is

$$(a^2 - 2xy - y^2)dx - (x + y)^2dy = 0$$

$$\Rightarrow a^2dx - 2xydx - y^2dx - x^2dy - 2xydy - y^2dy = 0$$

$$\Rightarrow a^2dx - y^2dy - d(x^2y) - d(xy^2) = 0$$

Now integrated both side, we have

$$a^2x - \frac{y^3}{3} - x^2y - xy^2 = c$$

but given that $y(0) = 1$ then

$$0 - \frac{1}{3} - 0 - 0 = c$$

i..e $c = -\frac{1}{3}$

then the required solution is

$$a^2x - x^2y - xy^2 - \frac{y^3}{3} = -\frac{1}{3}$$

$$3a^2x - 3x^2y - 3y^2x - y^3 + 1 = 0$$

33.(B,C) We know that square of standard normal variate becomes χ^2 -variate with and sum of two χ^2 -variate is again χ^2 -variate with 2-degree of freedom if both variate have same degree of freedom 1.

i.e. $X_1^2 + X_2^2 \sim \chi_{(2)}^2$

So, $P(X_1^2 + X_2^2 \leq 1) = P(\chi_{(2)}^2 < 1)$

$$P(\chi_{(2)}^2 \leq 1) = \int_0^1 \frac{1}{2^{2/2} \sqrt{2/2}} e^{-x/2} x^{\frac{2}{2}-1} dx = \int_0^1 \frac{1}{2} e^{-x/2} dx = \left[\frac{1 - e^{-x/2}}{2 \cdot (-1/2)} \right]_0^1 = [-e^{-1/2} + e^{-0}]$$

$$= 1 - \frac{1}{\sqrt{e}} \text{ or } \frac{\sqrt{e} - 1}{\sqrt{e}} = \frac{e - \sqrt{e}}{e}$$

34.(A,C,D) The critical region is given as

$$W = \{x : \bar{x} \geq K\}$$

where $\alpha = P\{x \in W / H_0\} = P\{\bar{x} \in k / \mu = 10\} \dots (1)$

Since $X \sim N(\mu, 1) \Rightarrow \bar{X} \sim N\left(\mu, \frac{1}{10}\right)$ [as $n = 10$]

Under H_0 , $\bar{X} \sim N\left(8, \frac{1}{10}\right) \Rightarrow Z = \frac{\bar{X} - E(\bar{X})}{\sigma_{\bar{x}}} = \frac{\bar{X} - 8}{1/\sqrt{10}} = \sqrt{10}(\bar{X} - 8)$

again under H_0 , $\bar{X} = K$ So, $Z = \sqrt{10}(k - 8)$

Now $P\{Z > \sqrt{10}(k-8)\} = 0.05$

$$\Rightarrow \sqrt{10}(k-8) = 1.65 \text{ [By the normal table]}$$

$$\Rightarrow K = 8 + 0.52$$

$$\Rightarrow K = 8.52$$

35.(A,B,C) $L = \prod_{i=1}^n \theta x_i^{\theta-1} = \theta^n (x_1 x_2 \dots x_n)^{\theta-1}$,

So $\frac{L_{H_1}}{L_{H_0}} = \frac{(-2)^n (x_1 x_2 \dots x_n)^1}{(1)^n (x_1 x_2 \dots x_n)^0} = 2^n (x_1 x_2 \dots x_n) \geq k = (x_1 x_2 \dots x_n)^{1/n} \geq \frac{k^{1/n}}{2}$

$$= (x_1 x_2 \dots x_n)^{1/n} \geq k_1 \quad \left[\text{where } k_1 = \frac{k^{1/n}}{2} \right]$$

36.(A,D) Since $\frac{E}{F}$ and $\frac{\bar{E}}{F}$ are complementary events.

Therefore, $P\left(\frac{E}{F}\right) + P\left(\frac{\bar{E}}{F}\right)$

$$\Rightarrow \frac{P(E \cap F)}{P(F)} + \frac{P(\bar{E} \cap F)}{P(F)} = \frac{P(E \cap F) + P(\bar{E} \cap F)}{P(F)} = 1.$$

$$1 - P\left(\frac{\bar{E}}{F}\right) - P\left(\frac{E}{F}\right) = 0$$

37.(A,B) E_1 : coin is fair, E_2 : coin is biased, a second toss shows tail.

$$P(E_1/A) = \frac{P(A/E_1)P(E_1)}{P(A/E_1)P(E_1) + P(A/E_2)P(E_2)} = \frac{\frac{m}{N} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{m}{N} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{N-m}{N} \cdot \frac{2}{3} \cdot \frac{1}{3}} = \frac{9m}{9m + 8N - 8m}$$

$$= \frac{9m}{8N+m} = \frac{18m^2}{16Nm+2m^2}$$

38.(A,B,D) We know that if $x \sim G(\lambda)$ distribution then m.g.f. of X is given as

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x) dx = \frac{1}{\sqrt{\lambda}} \int_0^{\infty} e^{tx} \cdot e^{-x} x^{\lambda-1} dx = \frac{1}{\sqrt{\lambda}} \int_0^{\infty} e^{-(1-t)x} x^{\lambda-1} dx$$

$$= \frac{\sqrt{\lambda}}{(1-t)^{\lambda}} \cdot \frac{1}{\sqrt{\lambda}}, \quad |t| < 1 = \frac{1}{(1-t)^{\lambda}}$$

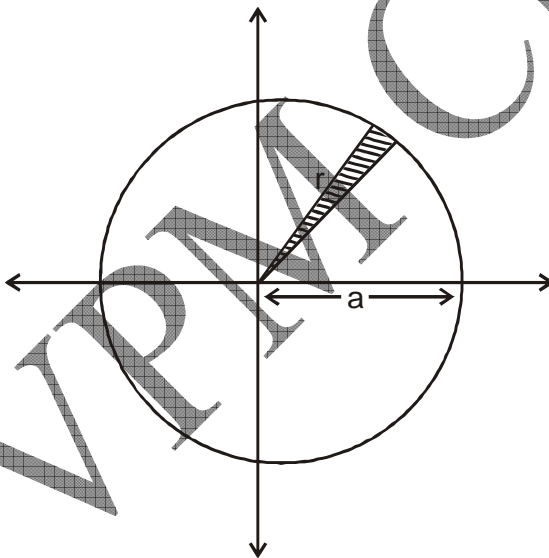
with comparison of our result we can say that

$$\lambda = 5 \quad \text{So } X \sim G(5)$$

39. (A,B,D) Putting $x = r \cos\theta$, $y = r \sin\theta$ and $dx dy = r dr d\theta$ in given integral

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2} dx dy = \int_0^a \int_0^{\frac{\pi}{2}} r^2 \sin^2 \theta \cdot r \cdot r dr d\theta$$

Here limits are find from the figure



$$= \int_0^a r^4 \left[\int_0^{\frac{\pi}{2}} \frac{1 - \cos 2\theta}{2} d\theta \right] dr = \int_0^a r^4 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/2} dr = \int_0^a r^4 \left[\frac{\pi}{4} - 0 \right] dr = \frac{\pi}{4} \left[\frac{r^5}{5} \right]_0^a = \frac{\pi a^5}{20}$$

40. (B,D) Given differential equation can be written as

$$(1 - x^2) \frac{dy}{dx} + xy = xy^2(1 - x^2)$$

$$\Rightarrow \frac{1}{y^2} \frac{dy}{dx} + \frac{x}{1-x^2} \frac{y}{y^2} = x$$

$$\text{let } \frac{1}{y} = t \Rightarrow -\frac{1}{y^2} \frac{dy}{dx} = \frac{dt}{dx}$$

$$\text{So, } -\frac{dt}{dx} + \frac{x}{1-x^2} t = x$$

$$\text{I.F. } e^{-\int \frac{x}{1-x^2} dx} \quad \text{let } 1 - x^2 = v \quad \Rightarrow -2x dx = dv \text{ or } -x dx = \frac{+dv}{2}$$

$$\therefore \text{I.F.} = e^{\int \frac{dv}{2v}} = e^{\frac{1}{2} \log v} = \sqrt{1-x^2}$$

then the general solution of equation is

$$t \cdot \sqrt{1-x^2} = \int -x \sqrt{1-x^2} dx + c$$

$$\text{again let } 1 - x^2 = w$$

$$\Rightarrow -x dx = dw/2$$

$$\Rightarrow \frac{1}{y} \sqrt{1-x^2} = \frac{1}{2} \int \sqrt{w} dw + c$$

$$\Rightarrow \frac{1}{y} \sqrt{1-x^2} = \frac{1}{2} \left[\frac{2}{3} w^{3/2} \right] + c$$

$$\Rightarrow \frac{1}{y} \sqrt{1-x^2} = \frac{1}{3} (1-x^2)^{3/2} + c$$

$$\Rightarrow \sqrt{1-x^2} = y \left[c + \frac{1}{3}(1-x^2)^{3/2} \right]$$

using initial condition $y(0) = 1$ we have

$$1 = 1 \left[c + \frac{1}{3} \right]$$

i.e. $c = \frac{2}{3}$

So, general solution is $\sqrt{1-x^2} = y \left[\frac{2}{3} + \frac{1}{3}(1-x^2)^{3/2} \right] = \frac{3}{2} \sqrt{1-x^2} = y \left[1 + \frac{(1-x^2)^{3/2}}{2} \right]$ **Ans.**

41. 4.06

The unbiased estimators of the population mean (μ) and the population variance (σ^2) are

$$\bar{x} = \frac{\sum x_i}{n} \quad \text{and} \quad S^2 = \frac{\sum (x_i - \bar{x})^2}{(n-1)} \text{ resp.}$$

Here $n = 5$ and $\bar{x} = \frac{\sum x_i}{n} = \frac{95}{5} = 19$

and, $\sum (x_i - \bar{x})^2 = (-5)^2 + 0^2 + (-2)^2 + 1^2 + 6^2 = 25 + 4 + 1 + 36 = 66$

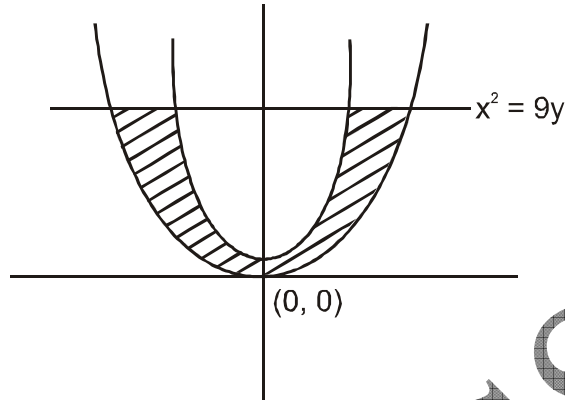
Hence, $s^2 = \frac{66}{4} = 16.5$

$$\Rightarrow s = \sqrt{16.5} = 4.06$$

Hence the estimators of σ is 4.06.

42. 9.426

Both curves are symmetric about y-axis. It is advisable to integrate about y-axis



$$x = \pm \frac{\sqrt{y}}{2} \quad x = \pm 3 \sqrt{y}$$

$$2 \int_0^5 \left(3\sqrt{y} - \frac{1}{2}\sqrt{y} \right) dy = 5 \int_0^2 \sqrt{y} dy = \frac{20\sqrt{2}}{3} = 9.426$$

43. 7

since A has 9 columns, $(\text{rank } A) + 2 = 9$, and hence rank A = 7.

44. 2

Since $X \sim N(\mu, 81)$ and $Y \sim N(\mu, 144)$

then $X + 2Y \sim N(3\mu, 657)$ and $2X + Y \sim N(\mu, 387)$

Now $P(X + 2Y \leq 3) = P(2X + Y \geq 4)$

$$\Rightarrow P\left(Z \leq \frac{3-3\mu}{\sqrt{657}}\right) = P\left(Z \leq \frac{4-\mu}{\sqrt{387}}\right)$$

[using $P(z \leq a) = P(z \geq b) \Rightarrow a = -b$]

$$\text{then } -\left(\frac{3-3\mu}{\sqrt{657}}\right) = \left(\frac{4-\mu}{\sqrt{387}}\right)$$

$$\Rightarrow \frac{3\mu - 3}{\sqrt{657}} = \frac{4 - \mu}{\sqrt{387}}$$

$$\Rightarrow \frac{3\mu - 3}{25.63} = \frac{4 - \mu}{19.67}$$

$$\Rightarrow 59\mu - 59 = 102.52 - 25.63\mu$$

$$\Rightarrow 84.63 \mu = 161.52$$

$$\Rightarrow \mu = 1.908$$

2 (approximate)

45. 0.375

We have, for instance,

$$\begin{aligned} P\left(0 < x_1 < \frac{3}{4}, \frac{1}{3} < x_2 < 2\right) &= \int_{1/3}^2 \int_0^{3/4} f(x_1, x_2) dx_1 dx_2 = \int_{1/3}^1 \int_0^{3/4} 6x_1^2 x_2 dx_1 dx_2 + \int_1^2 \int_0^{3/4} 0 dx_1 dx_2 \\ &= \frac{3}{8} + 0 = \frac{3}{8} = 0.375 \end{aligned}$$

46. 0.33

Let X_1 and X_2 have the joint pdf.

$$f(x_1, x_2) = \begin{cases} 6x_2 & 0 < x_2 < x_1 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then the marginal pdf of X_1 is

$$f_1(x_1) = \int_0^{x_1} 6x_2 dx_2 = 3x_1^2, \quad 0 < x_1 < 1,$$

zero elsewhere. The conditional pdf of X_2 , given $X_1 = x_1$, is

$$f_{2|1}(x_2 | x_1) = \frac{6x_2}{3x_1^2} = \frac{2x_2}{x_1^2}, \quad 0, x_2 < x_1,$$

zero elsewhere, where $0 < x_1 < 1$. The conditional mean of X_2 , given $X_1 = 1/2$, is

$$E\left(X_2 | x_1 = \frac{1}{2}\right) = \int_0^{x_1} x_2 \left(\frac{2x_2}{x_1^2}\right) dx_2 = \frac{2}{3} x_1, \quad 0 < x_1 < 1.$$

i.e. $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3} = 0.33$

47. 2

$$x_1 = \sqrt{2}, \quad x_2 = \sqrt{2x_1} = \sqrt{2\sqrt{2}}$$

$$\therefore \sqrt{2} > 1 \quad \therefore 2\sqrt{2} > 2 \Rightarrow \sqrt{2\sqrt{2}} > \sqrt{2}$$

$$\Rightarrow x_2 > x_1 \text{ or } x_1 < x_2$$

Now $x_{r-1} < x_r \Rightarrow \sqrt{2x_{r-1}} < \sqrt{2x_r}$

$$\Rightarrow x_r < x_{r+1}$$

by the mathematical induction

$$x_n < x_{n+1} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \{x_n\} \text{ monotonically increasing} \quad \dots(1)$$

again, $x_1 < 2$ Let $x_r < 2$

$$\Rightarrow \sqrt{2x_r} < \sqrt{2\sqrt{2}} = 2$$

$$\Rightarrow x_{r+1} < 2$$

again by the mathematical induction

$$x_n < 2 \quad \forall n \in \mathbb{N} \quad \dots(2)$$

Thus sequence $\{x_n\}$ is bounded

(1) and (2) \Rightarrow sequence $\{x_n\}$ is convergent

Let $\lim x_n = \xi$

$$\Rightarrow \lim x_n = \lim x_{n+1} = \xi$$

$$\therefore x_{n+1} \sqrt{2x_n} = \Rightarrow \lim x_{n+1} = \sqrt{2 \lim x_n}$$

$$\Rightarrow \xi = \sqrt{2\xi} \Rightarrow \xi^2 - 2\xi = 0$$

$$\Rightarrow \xi = 0 \text{ or } \xi = 2 \text{ but } x \neq 0 \text{ so } \xi = 2.$$

48. 0.1

The probability that the first ball drawn is black is $P_r \{E_1\} = 2/(3+2) = \frac{2}{5}$. The probability

that the second ball drawn is black, given that the first ball drawn was black, is $P_r \{E_2 | E_1$

$\} = 1/(3+1) = \frac{1}{4}$. Thus the probability that both balls drawn are black is

$$\Pr\{E_1 E_2\} = P_r\{E_2 | E_1\} = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10} = 0.1$$

49. **0.5049**

We have $P(\text{head on exactly 50 coins}) = P(\text{head on exactly 51 coins})$

$$\Rightarrow {}^{100}C_{50} \times p^{50} (1-p)^{50} = {}^{100}C_{51} \times p^{51} (1-p)^{49}$$

$$\Rightarrow p = \frac{51}{101} = 0.5049$$

50. **4**

Binomial distribution is defined as

$$B(n; p; q) = {}^{10}C_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{10-x}; \quad x = 0, 1, \dots, 10$$

if we compare it by the binomial distribution then we can say that

$$n = 10, p = \frac{2}{5} \quad \text{and} \quad q = \frac{3}{5}$$

and mean of Binomial distribution is

$$E(x) = np = 10 \times \frac{2}{5} = 4$$

51. **12**

Since $X \sim P(1)$ and $Y \sim P(3)$ as 1, 3 are means of random variable X and Y resp. and we know that mean of random variable which follows Poisson distribution is equal to parameter λ .

Now $E(X) = \lambda_1 = 1$

then $\text{Var}(X) = \lambda_1 = 1$

again $E(Y) = \lambda_2 = 3$

then $\text{Var}(Y) = \lambda_2 = 3$

So, $\text{Var}(3X + Y) = 9\text{Var}(X) + \text{Var}(Y)$ [$\because X, Y$ are independent]

$$= 9(1) + 3 = 9 + 3 = 12$$

52. 0.375

Let E denote the event that a six occurs and A the event that it is a six. We have $P(E) =$

$1/6$, $P(E') = 5/6$, $P(A/E) = \frac{3}{4}$ and $P(A/E') = 1/4$. By Baye's theorem

$$P(E/A) = \frac{P(E).P(A/E)}{P(E).P(A/E) + P(E').P(A/E')} = \frac{\left(\frac{1}{6}\right)\left(\frac{3}{4}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{4}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{4}\right)} = \frac{3}{8} = 0.375$$

53. 0.25

Let the lengths of three parts of the rod be x , y and $l - (x + y)$.

Then $x > 0$, $y > 0$ and $l - (x + y) > 0$

i.e., $x + y < l$ or $y < l - x$

Since, in a triangle, the sum of any two sides is greater than third side, so

$$x + y > l - (x + y)$$

$$\Rightarrow y > \frac{\ell}{2} - x$$

$$x + \ell - (x + y) > y$$

$$\Rightarrow y < \frac{\ell}{2}$$

$$y + \ell - (x + y) > x$$

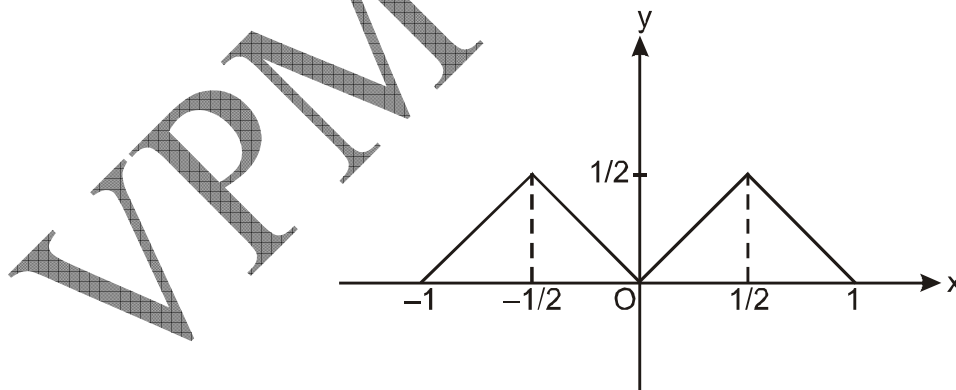
$$\Rightarrow x < \frac{\ell}{2}$$

$$\Rightarrow \frac{\ell}{2} - x < y < \frac{\ell}{2} \text{ and } 0 < x < \frac{\ell}{2}$$

$$\text{So, required probability} = \frac{\int_0^{\ell/2} \int_{\ell/2-x}^{\ell/2} dy dx}{\int_0^{\ell} \int_0^{\ell-x} dy dx} = \frac{\int_0^{\ell/2} \left\{ \frac{\ell}{2} - \left(\frac{\ell}{2} - x \right) \right\} dx}{\int_0^{\ell} (\ell - x) dx} = \frac{\int_0^{\ell/2} x dx}{\int_0^{\ell} (\ell - x) dx} = \frac{\frac{\ell^2}{8}}{\frac{\ell^2}{2}} = \frac{1}{4} = 0.25$$

54. 0.5

Graph of $f(x) = \min(|x|, |x-1|, |x+1|)$



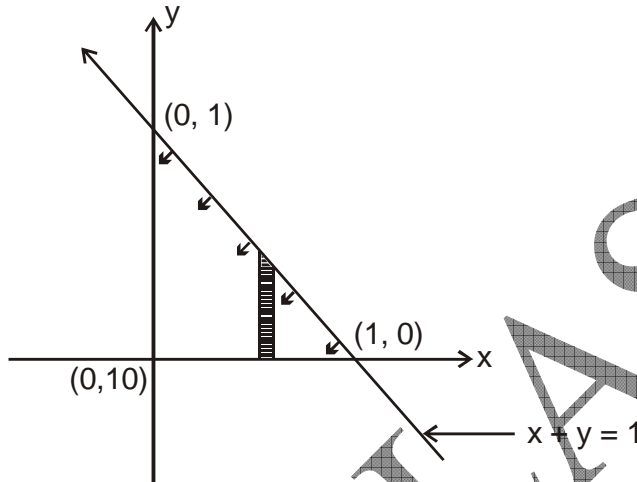
$$\text{Required area} = 2 \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{2} \text{ sq unit} = 0.5 \text{ sq unit}$$

55. 0.0416

From the figure it is clear that the limits of x and y are as x varies from 0 to 1 and y from 0 to

$$1 - x$$

then



$$\iint xy \, dx \, dy = \int_0^1 \int_0^{1-x} xy \, dx \, dy = \int_0^1 \left[\frac{xy^2}{2} \right]_0^{1-x} dx = \int_0^1 \frac{x}{2} [(1-x)^2 - 0] dx = \frac{1}{2} \int_0^1 x(1-x)^2 dx$$

$$= \frac{1}{2} \int_0^1 [x - 2x^2 + x^3] dx = \frac{1}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1 = \frac{1}{2} \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = \frac{1}{24} = 0.0416$$

56. 3

Given differential equation is

$$\frac{d^2y}{dx^2} + 9y = 0 \Rightarrow (D^2 + 9)y = 0$$

$$\Rightarrow (D + 3i)(D - 3i)y = 0$$

$$\text{So, } y(x) = c_1 \cos 3x + c_2 \sin 3x$$

which shows that two L.I. solutions of differential equation is $\cos 3x$ and $\sin 3x$.

$$\text{Now } W(x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3\cos^2 3x + 3\sin^2 3x = 3$$

i.e. $W(x) = 3$

which shows that wronskian does not depends on x.

i.e. $w(1) = 3 = w(s)$.

57. 1.414

Given limit is

$$\lim_{x \rightarrow \infty} \frac{2x^{1/2} + 3x^{1/3} + 4x^{1/4} + \dots + nx^{1/n}}{(2x-3)^{1/2} + (2x-3)^{1/3} + \dots + (2x-3)^{1/n}}$$

if we divides numerator and denominator by then we get

$$\lim_{x \rightarrow \infty} \frac{2 + 3x^{-1/6} + \dots + nx^{1/n-1/2}}{\sqrt{2} \left[\left(1 - \frac{3}{2x}\right)^{1/2} + \left(1 - \frac{3}{2x}\right)^{1/3} + \dots + \left(1 - \frac{3}{2x}\right)^{1/n} \right]} \text{ where } n > 0$$

after applying limit we get the required result as

$$\frac{2}{\sqrt{2}} = \sqrt{2} = 1.414$$

58. 0

We know that taylor's expansion is given as about origin is

$$f(0+x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

Given $f(0) = -1$

$$f'(0) = 1 + \frac{1}{f(0)} = 1 - 1 \quad [\because f(0) = -1]$$

$$= 0$$

$$f''(0) = \frac{-f(0)}{[f(0)]^2} = \frac{0}{[-1]^2} = \frac{0}{1} = 0$$

Thus, the coefficient of x^2 in the Taylor's expansion of f about zero is $\frac{f''(0)}{2!} = \frac{0}{2!} = 0$

59. 0.5

Given that q is parameter of binomial distribution i.e. $E(X) = n\theta$

Now

$$E = \left[\frac{X + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}} \right] = \frac{1}{n + \sqrt{n}} E \left[X + \frac{1}{2}\sqrt{n} \right] = \frac{1}{n + \sqrt{n}} \left[E(X) + \frac{1}{2}\sqrt{n} \right] = \frac{1}{n + \sqrt{n}} \left[n\theta + \frac{1}{2}\sqrt{n} \right]$$

By the observation we can say that

ie we put $\theta = \frac{1}{2}$ then

$$E \left[\frac{X + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}} \right] = \frac{1}{2} = \theta$$

which shows that the minimax estimator is unbiased estimator $\theta = \frac{1}{2} = 0.5$.

60. 0.91

$$M_X(t) = e^{4(e^t - 1)}$$

which is the mgf of Poisson distribution with parameter $\lambda = 4$

Now $E(x) = \mu = \lambda = 4$

and $\text{Var}(x) = \sigma^2 = \lambda = 4$ i.e. $\sigma = 2$

then $P(\mu - 2\sigma < x < \mu + 2\sigma) = P(\mu - 4 < x < \mu + 4) = P(0 < x < 8) = \sum_{x=1}^7 P(X = x)$

$$= \sum_{x=1}^7 \frac{e^{-4} (4)^x}{x!} = e^{-4} \sum_{x=1}^7 \frac{(4)^x}{x!} = 0.018 \left[\frac{4^1}{1!} + \frac{4^2}{2!} + \dots + \frac{4^7}{7!} \right] = 0.018[50.806] = 0.91$$

VPM CLASSES