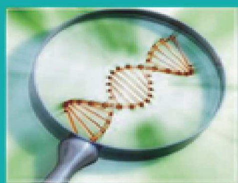

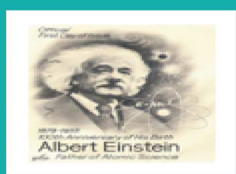

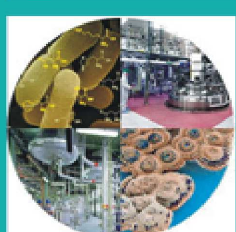

IIT-JAM - MATHEMATICAL STATISTICS

MOCK TEST PAPER(According to new pattern)

- Attempt ALL the 60 questions.
- SECTION-A Consists of 30 questions. These questions are Multiple Choice Questions (MCQs),First 20 questions carries one marks for each and remaining 10 questions carries two marks for each.
- Section-B Consists of 10 questions. These questions are Multiple Select Questions (MSQs), each question carries two marks.
- Section-C Consists of 20 Numerical Answer Type (NAT) questions each question carries two marks. For each NAT type question, the answer is a signed real number.
- In Section A, for all 1 mark questions,1/3 marks will be deducted for each wrong answer and for all 2 marks questions,2/3 marks will be deducted for each wrong answer.There is no negative marking in Section B and Section C.

- ***Total marks : 100***
- ***Duration of test : 3 Hours***

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SECTION-A (1-30) : MULTIPLE CHOICE QUESTIONs (MCQs)

1. A coin is tossed m + n times, where $m \geq n$. The probability of getting at least m consecutive heads is

(A) $\frac{n+1}{2^{m+1}}$

(B) $\frac{n+2}{2^{m+1}}$

(C) $\frac{m+2}{2^{n+1}}$

(D) None of these

2. If a variable takes values 0, 1, 2, ...n with frequencies $q^n, \frac{n}{1}q^{n-1}p, \frac{n(n-1)}{1.2}\ q^{n-2}p^2, ...p^n$, where p + q = 1, then the mean is

(A) np

(B) nq

(C) n(p + q)

(D) None of these

3. Let $f(x) = \sin^3 x + \lambda \sin^2 x, -\frac{\pi}{2} < x \frac{\pi}{2} <$. Find the intervals in which λ should lie in order that f(x) has exactly one minimum and exactly one maximum.

(A) $\left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$

(B) $\left(\frac{-3}{2}, 0\right) \cup \left(\frac{3}{2}, \infty\right)$

(C) $\left(-\infty, \frac{-3}{2}\right) \cup \left(0, \frac{3}{2}\right)$

(D) $\left(\frac{-3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$

4. If a + b + c = 0, then one of the solution of

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0 \text{ is}$$

(A) x = a + b – c

(B) x = 0

(C) x = a – b + c

(D) x = b + c – a

5. Let Ax = b be a homogeneous system of linear equation. The augmented matrix. [A : b] is given by

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 4 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

Which of the following statements is true ?

(A) The system has no Solution

(B) The system has unique Solution

(C) The system has infinitely many solutions

(D) The system has finite number of solutions.

6. A coin is tossed repeatedly until a head appears. Find the expected number of tosses required to obtain the first heads.

(A) $\frac{p}{q}$

(B) $\frac{1}{q}$

(C) $\frac{1}{q^2}$

(D) $\frac{1}{p}$

7. Let $X_1, X_2, ...X_n$ be a sequence of mutually independent random variables with common distribution. Suppose X_k assumes only positive integral values and $E(X_k) = a$ exists; $K = 1, 2, ...n$. let $S_n = X_1 + X_2 + ... + X_n$ then find $E,\left(\frac{S_m}{S_n}\right)$ for $1 \le m \le n$

(A) $\frac{n}{m}$

(B) $\frac{1}{m}$

(C) $\frac{m}{n}$

(D) $\frac{1}{n}$

8. Let X be a random variables with probability function

$$P(x = \pm 2^x) = \begin{cases} \dfrac{e^{-1}}{2x!}, & x = 0, 1, 2.... \\ 0 & \text{otherwise.} \end{cases}$$

then find the moment generating function of X.

(A) $M_X(t) = \int_a^\theta \frac{e^{tx}}{x^{\theta+1}} dx$

(B) $M_X(t) = \theta \int_a^\infty \frac{e^{tx}}{x^{\theta+1}} dx$

(C) $M_X(t) = \theta a^\theta \int_a^\theta \frac{e^{tx}}{x^{\theta+1}} dx$

(D) does not exists

9. If X and Y are independent random Variables, and A = { x ; a < x < b},

B = {y : c < y < d}

be any two events, then find $P(A \cap B)$?

(A) P(A) + P(B)

(B) P(A).P(B)

(C) P(A)

(D) P(B)

10. Let the joint probability density function of a two dimensional random variable (X, Y) be;

$$f(x, y) = \begin{cases} x + y, 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 \quad , \end{cases}$$

then find the conditional density function of Y given X = x is

(A) $\frac{2x}{2x+1}; 0 \le y \le 1$

(B) $\frac{2(x+y)}{2x+1}; 0 \le y \le 1$

(C) $\frac{2y}{2x+1}; 0 \le y \le 1$

(D) $\frac{x+y}{2x+1}; 0 \le y \le 1$

11. If f(x) = $\begin{cases} ax+b, \text{ if } x \le \frac{\pi}{4} \\ \cos x, \text{ if } x > \frac{\pi}{4} \end{cases}$ then f(x) is differentiable at $x = \frac{\pi}{4}$, if

(A) $a = -\frac{1}{\sqrt{2}}, b = \frac{1+\frac{\pi}{4}}{\sqrt{2}}$

(B) $a = \frac{1}{\sqrt{2}}, b = \frac{1+\frac{\pi}{4}}{\sqrt{2}}$

(C) $a = b = \frac{1}{\sqrt{2}}$

(D) $a = b = -\frac{1}{\sqrt{2}}$

12. The area of the region enclosed by curves $x^2 + y^2 = a^2, x + y = a$ (in first quadrant)).

(A) $a^2(\pi - 2)$

(B) $\frac{a^2}{2}(\pi - 2)$

(C) $\frac{a^2}{3}(\pi - 2)$

(D) $\frac{a^2}{4}(\pi - 2)$.

13. Find the M.L.E of α and β for a random sample from the exponential population $f(x;\alpha,\beta) = y_0 e^{-\beta(x-\alpha)},\ \alpha \le x \le \infty,\ \beta \ge 0$ and y_0 being a constant.

(A) $\hat{\alpha} = \overline{x} - x_{(1)}, \hat{\beta} = \frac{1}{\overline{x} - x_{(1)}}$

(B) $\hat{\alpha} = \overline{x}, \hat{\beta} = \frac{1}{\overline{x} - x_{(1)}}$

(C) $\hat{\alpha} = \frac{1}{\overline{x} - x_{(1)}}, \hat{\beta} = x_{(1)}$

(D) $\hat{\alpha} = x_{(1)}, \hat{\beta} = \frac{1}{\overline{x} - x_{(1)}}$

14. For the distribution with density function $f(x,\theta) = \frac{x}{1+\theta}; 0 \le x \le 1$ then the estimate of θ by method of moments

(A) $\frac{1}{3M_1'} - 1$

(B) $3M_1' - 1$

(C) $\frac{1}{\sqrt{3M_1^{'}-1}}$

(D) $M_1^{'}$

15. Find the maximum value of the function $f(x) = x^2 e^{-x}$

(A) $\frac{4}{e^2}$

(B) $\frac{4}{e^3}$

(C) $\frac{2}{e^2}$

(D) $\frac{2}{e^3}$

16. Solve the differential equation

$$y - x\left(\frac{dy}{dx}\right) = a\left[y^2 + \frac{dy}{dx}\right], \qquad y(0) = 1$$

(A) $\frac{x+a}{a} = y(1 + x)$

(B) $\frac{x+a}{a} = y(x - 1)$

(C) $\frac{x-a}{a} = x(1 + y)$

(D) $\frac{x-a}{a} = x(y - 1)$

17. Suppose B = {$\mathbf{b_1}$, $\mathbf{b_2}$} is a basis for V and C = {c_1, c_2, c_3} is a basis for W. let

T : V $\to$ W be a linear transformation with the property that

$$T(\mathbf{b_1}) = 3c_1 - 2c_2 + 5c_3 \quad \text{and} \quad T(\mathbf{b_2}) = 4c_1 + 7c_2 - c_3$$

Find the matrix M for T relative to B and C.

(A) $\begin{bmatrix} 3 & -2 & 5 \\ 4 & 7 & -1 \end{bmatrix}$

(B) $\begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 4 & 7 & -1 \\ 3 & -2 & 5 \end{bmatrix}$

(D) $\begin{bmatrix} 4 & 3 \\ 7 & -2 \\ -1 & 5 \end{bmatrix}$

18. If X ~ N(0, 1) then find the correlation

coefficient of (x, y) where $y = a + bx + cx^2$.

(A) $\dfrac{a}{\sqrt{b^2 + c^2 - ab}}$

(B) $\dfrac{b}{\sqrt{a^2 + c^2 - ac}}$

(C) $\dfrac{b}{\sqrt{b^2 + 2c^2 - ac}}$

(D) $\dfrac{c}{\sqrt{a^2 + 2b^2 - ac}}$

19. Let X_1 and X_2 have the joint pdf

$$f_{x_1,x_2}(x_1, x_2) = \begin{cases} 10x_1x_2^2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

if $Y_1 = \dfrac{X_1}{X_2}$ and $Y_2 = X_2$ then find the marginal pdf of Y_1.

(A) $f_{y_1}(y_1) = 2y_1, 0 < y_1 < 1$

(B) $f_{y_1}(y_1) = 5y_1^4, 0 < y_1 < 1$

(C) $f_{y_1}(y_1) = \dfrac{y_1}{2}, 0 < y_1 < 1$

(D) $\dfrac{y_1}{2} = \dfrac{y_1^4}{5}, 0 < y_1 < 1$

20. If x be a random variable such that

$$E(x^m) = \frac{(m+3)!}{3!} 3^m, \quad m = 1, 2, 3, \ldots\ldots$$

and the mgf of x is given by

$$M(t) = 1 + \frac{4!3}{3!1!} t + \frac{5!3^2}{3!2!} t^2 + \frac{6!3^3}{3!3!} t^3 + \ldots\ldots$$

then the random variable X follows the distribution

(A) $\gamma(3, 4)$

(B) $\gamma(4, 3)$

(C) $\beta(3, 4)$

(D) $\beta(4, 3)$

21. Let the Random variable x have the p.m.f.

$$f(x, \theta) = \frac{e^{-\theta}\theta^{x}}{x!}, \qquad x = 0, 1, 2, \ldots .$$

then find the sufficient statistic for θ.

(A) $\hat{\theta} = \sum_{i=1}^{n} x_i$

(B) $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i$

(C) $\hat{\theta} = \prod_{i=1}^{n} x_i$

(D) $\hat{\theta} = \frac{1}{n}\prod_{i=1}^{n} x_i$

22. A sample of 900 members has a mean 3.4cms. and s.d. 261 cms. Is the sample from a large population of mean 3.25 cms. and s.d. 2.61 cms ?

(A) Yes

(B) No

(C) datas are incomplete

(D) cannot be determined

23. $\lim_{x \to \frac{\pi}{4}} \dfrac{\int_{2}^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}}$ equals

(A) $\frac{8}{\pi} f(2)$

(B) $\frac{2}{\pi} f(2)$

(C) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$

(D) 4f(2).

24. Three six-faced dice are thrown together. The probability that the sum of the numbers operating on the dice ,$k(3 \le k \le 8)$, is

(A) $\frac{(k-1)(k-2)}{432}$.

(B) $\frac{(k-1)(k^2-2)}{42}$.

(C) $\frac{(k-3)(k-5)}{432}$.

(D) $\frac{(k-6)(k-2)}{432}$.

25. Let the joint probability density function of continuous random variables x_1 and x_2 be given by

$$f(x_1, x_2) = \begin{cases} 2, & 0 < x_1 < x_2 < 1 \\ 0, & \text{otherwise} \end{cases}$$

then find the $E(x_1 \mid x_2)$, $Var(x_1 \mid x_2)$

(A) $\frac{x_2}{2}$, $0 < x_2 < 1$, , $\frac{x_2^2}{12}$ $0 < x_2 < 1$.

(B) $\frac{x_2}{3}$, $0 < x_2 < 1$, $\frac{x_2^2}{2}$, $0 < x_2 < 1$.

(C) $\frac{x_2}{12}$, $0 < x_2 < 1$, $\frac{x_2^2}{42}$, $0 < x_2 < 1$.

(D) None of these

26. Solution of the differential equation

$$(2x \cos y + y^2 \cos x)dx + (2y \sin x - x^2 \sin y)dy = 0 \text{ is}$$

(A) $x^2 \cos y + y^2 \sin x = c$

(B) $x^2 \cos y + z^2 \sin x = c$

(C) $y^2 \cos y + x^2 \sin x = c$

(D) None of these

27. Three six-faced dice are thrown together. The probability that the sum of the numbers operating on the dice is

$k(3 \le k \le 8)$, is

(A) $\frac{k^2}{432}$

(B) $\frac{k(k-1)}{432}$

(C) $\frac{(k-1)(k-2)}{432}$

(D) $\frac{k(k-1)(k-2)}{432}$

28. If X is a $G\left(5,\frac{1}{3}\right)$ random variables, then P(X > G) equals

(A) e^{-1}

(B) e^{-2}

(C) $\frac{e^{-1}}{3}$

(D) $\frac{e^{-2}}{3}$

29. Let f be four times differentiable function on $\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$ such that $f'(x)=1+\left[f(x)\right]^3$. If f(0) = 1, then the coefficient of x^3 in Taylor's expansion of f about zero is

(A) 6

(B) 7

(C) 8

(D) 9

30. Find the value of $\lim_{x\to\infty}\dfrac{\int_0^{2x} te^{t^2}dt}{e^{4x^2}}$.

(A) ∞

(B) 2

(C) $\frac{1}{2}$

(D) 0

SECTION - B (31-40) : MULTIPLE SELECT QUESTIONs (MSQs)

31. Consider the function $f(x,y)=\begin{cases}\dfrac{xy\left(x^2-y^2\right)}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & \text{otherwise}\end{cases}$, then which of the following is/are false for the given function ?

(A) $f_{xy}(0, 0) = f_{yx}(0, 0) = 1$

(B) $f_{xy}(0, 0) = f_{yx}(0, 0) = -1$

(C) $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

(D) $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$

32. What is/are solution of the differential equation $(a^2 - 2xy - y^2)dx - (x + y)^2dy = 0$, $y(0) = 1$?

(A) $a^2x - x^2y - y^2x - \dfrac{y^3}{3} = -\dfrac{1}{3}$

(B) $a^2x - xy^2 - x^2y + \dfrac{y^3}{3} = +\dfrac{1}{3}$

(C) $3a^2x - 3x^2y - 3y^2x - y^3 + 1 = 0$

(D) $a^2x - xy(x + y) + \frac{xy}{3} = 0$

33. If X_1 and X_2 are i.i.d. N(0, 1) random variables, then $P\left(X_1^2 + X_2^2 \le 1\right)$

(A) $\frac{e-1}{e}$

(B) $\frac{e-\sqrt{e}}{e}$

(C) $\frac{\sqrt{e}-1}{\sqrt{e}}$

(D) $\frac{e^2-1}{e^2}$

34. Let X_1, X_2, ... X_{10} be a random sample from a N(μ, 1) distribution. For testing $H_0 : \mu = 8$ against $H_1 : \mu = 9$ the most powerful critical region is $\bar{X} \ge K,$ then which of the following value of k is/are incorrect for the size of the test is 0.05 ?

(A) K = 14.5

(B) K = 8.52

(C) K = 17.04

(D) K = 8.00

35. Let X be a random variable with the probability density function

$$f(x \backslash \theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \theta > 0 \\ 0, & \text{otherwise} \end{cases}$$

then which of the following is/are not most powerful critical region (rejection region) for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ with size of critical region is α ?

(A) $\dfrac{x_1 + 2x_2 + ... + nx_n}{n} \geq k_1$

(B) $\dfrac{x_1 + x_2 + ... + x_n}{n} n \geq k_1$

(C) $(x_1, x_2 ..... x_n)^{1/n} \leq k_1$

(D) $(x_1\, x_2 ...... x_n)^{1/n} \geq k_1$

36. If $\bar{E}$ and $\bar{F}$ are the complementary events of events E and F respectively and if 0 < P (F) < 1, then

(A) $P\left(\dfrac{E}{F}\right) + P\left(\dfrac{\bar{E}}{F}\right) = 1$

(B) $P\left(\dfrac{E}{F}\right) + P\left(\dfrac{E}{\bar{F}}\right) = 1$

(C) $P\left(\dfrac{\bar{E}}{F}\right) + P\left(\dfrac{E}{\bar{F}}\right) = 1$

(D) $1 - P\left(\dfrac{\bar{E}}{F}\right) - P\left(\dfrac{E}{F}\right) = 0$

37. A box contains N coins, m of which are fair and the rest are baised. The probability of getting a head when a fair coin is tossed is $\frac{1}{2}$, while it is $\frac{2}{3}$ when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. Then the probability that the coin drawn is fair, is

(A) $\frac{9m}{8N+m}$

(B) $\frac{18m^2}{16Nm+2m^2}$

(C) $\frac{9m}{8m-N}$

(D) $\frac{9m}{8m+N}$

38. If the moment generating function of a random variable X is given as $\frac{1}{(1-t)^5}, |t|<1$ then random variable X does not follow ---

(A) Exponential distribution with parameter 5

(B) χ^2-distribution with degree of freedom 5

(C) gamma distribution with parameter 5

(D) Poisson distribution with parameter 5.

39. Which of the following is/are not equal to the value of integral $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2}\, dx\, dy$?

(A) $\frac{\pi a^2}{30}$

(B) $\frac{\pi a^2}{20}$

(C) $\frac{\pi a^2}{20}$

(D) $\frac{\pi a^2}{30}$

40. Find the general solution of the differential equation (1 – x2) $\frac{dy}{dx} + xy = xy^2\left(1-x^2\right), y(0) = 1$

(A) $y\sqrt{x^2-1} = \left[2-\frac{1}{3}\left(x^2-1\right)^{3/2}\right]$

(B) $\frac{3}{2}\sqrt{1-x^2} = y\left[1+\frac{\left(1-x^2\right)^{3/2}}{2}\right]$

(C) $\sqrt{-x^2+1} = \frac{y}{3}\left[2+\left(-x^2+1\right)^{3/2}\right]$

(D) $\sqrt{1-x^2} = y\left[\frac{2}{3}+\frac{1}{3}\left(1-x^2\right)^{3/2}\right]$

SECTION - C (41-60) : NUMERICAL ANSWER TYPE (NATs)

41. The following observations constitute a random sample from an unknown population. Find the estimator of the standard deviation of the population.

14, 19, 17, 20, 25

42. Find the area bounded by the parabolas $y = 4x^2$, $y = x^2/9$ and the straight line $y = 2$.

43. If a is 7 × 9 matrix with a two dimensional null space, then what is the Rank of A ?

44. If $X \sim N(\mu, 9^2)$ and $Y \sim N(\mu, 12^2)$ are independent and

if $P(X + 2Y \le 3) = P(2X - Y \ge 4)$, determine μ.

45. Let $f(x_1, x_2) = \begin{cases} 6x_1^2 x_2 & 0 < x_1 < 1, 0 < x_2 < 1 \\ 0 & \text{elsewhere,} \end{cases}$ be the pdf of two random variables X_1 and X_2 of the continuous type then find $P\left(0 < x_1 < \frac{3}{4}, \frac{1}{3} < x_2 < 2\right)$.

46. Let X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} 6x_2 \ , & 0 < x_2 < x_1 < 1 \\ 0 \ , & \text{otherwise} \end{cases}$$

then find $E\left(x_2 \mid x_1 = \frac{1}{2}\right)$.

47. Consider the sequence $\{x_n\}$, where $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$,then what is the value of ξ?

48. Suppose that a box contains 3 white balls and 2 black balls. Let E_1 be the event "first ball drawn is black" and E_2 the event "second ball drawn is black," where the balls are not replaced after being drawn. Here E_1 and E_2 are dependent events, then what will be the probability if both balls drawn are black?

49. One hundred Identical coins, each with probability p, of showing up a head, are tossed. If $0 < p < 1$ and if the probability of heads on exactly 50 coins is equal to that of heads on exactly 51 coins then then find the value of p ?

50. What is the mean of the binomial distribution $\binom{10}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{10-x}$; $x = 0, 1, 2, \ldots, 10$?

51. If X and Y are independent Poisson variates with means 1 and 3 respectively then find the value of variance of $3X + Y$?

52. A man is known to speak truth in 75% cases. If he throws an unbiased die and tells his friend that it is a six, then find the probability that it is actually a six.

53. A wire of length ℓ is cut into three pieces. What is the probability that the three pieces form a triangle?

54. Let the function defined by f(x) = min {|x|, |x – 1|, |x + 1| }

then find the area (in sq. unit) ,which is bounded by f(x) in [– 1, 1] and the x-axis.

55. If the region of the integration is $x + y \le 1, x \ge 0, y \ge 0$,then what will be the value of $\iint xy\, dx\, dy$?

56. If y_1 and y_2 are the two linearly independent solution of differential equation $\frac{d^{2y}}{dx^2} = 9y = 0$ with w(1) = +3 then find w(s).

57. What will be the value of limit $\lim_{x \to \infty} \frac{2x^{1/2} + 3x^{1/3} + 4x^{1/4} + \ldots\ldots + nx^{1/n}}{(2x-3)^{1/2} + (2x-3)^{1/3} + \ldots\ldots + (2x-3)^{1/n}}$?

58. Let f be a thrice differentiable function on $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$ such that f'(x) = 1 + [f(x)]–1. If f(0) = –1 then what will be the coefficient of x^2 in Taylor's expansion of f about zero ?

59. For which value of q the minimax estimator $\frac{X + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}}$ is unbiased estimator of binomial parameter ?

60. If the mgf of a r.v. x is $e^{4(e^t - 1)}$ then find the value of P(μ – 2σ < x < μ + 2σ).

ANSWER KEY

[redacted]

HINTS AND SOLUTION

1.(B) Here $P(H)=P(T)=\frac{1}{2}$ and $P(X)=1$, where X denotes head or tail.

If the sequence of m consecutive heads starts from the first throw, we have (HH.... m times) (XX ..... n times).

$\therefore$ Chance of this event $=\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}.......m\text{ times}=\frac{1}{2^m}$

m + 1 and subsequent throws may be head or tail since we are considering at least m consecutive heads. If the sequence of m consecutive heads starts from the second thrown, the first must be a tail and we have, the chance of this event $=\frac{1}{2}\cdot\frac{1}{2^m}=\frac{1}{2^{m+1}}$.

If the sequence of heads starts from (r + 1)th throw then the first (r – 1) throws may be head or tail but rth throw must be a tail and we have,

(XX.......(r – 1) times) T(HH...., n times)

(XX....... $n - \overline{m - r}$ times)

The chance of this event also $\frac{1}{2} \times \frac{1}{2^m} = \frac{1}{2^{m+1}}$

Since all the above events are mutually exclusive , so the required probability

$$= \frac{1}{2^m} + \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} + \ldots.n\,\text{times}\right) = \frac{1}{2^m} + \frac{n}{2^{m+1}} = \frac{n+2}{2^{m+1}}.$$

2.(A) The required mean is,

$$\bar{x} = \frac{0.q^n + 1.\frac{n}{1}q^{n-1}p + 2.\frac{(n)(n-1)}{2!}q^{n-2}p^2 + \ldots\ldots n.p^n}{q^n + \frac{n}{1}q^{n-1}p + \frac{n(n-1)}{2}q^{n-2}p^2 + \ldots.. + p^n}$$

$$= \frac{0.{}^nC_0q^np^0 + 1.{}^nC_1q^{n-1}p + \ldots.. + n.{}^nC_nq^0p^n}{nC_0q^np^0 + {}^nC_1p^{n-1}p^1 + \ldots\ldots + {}^nC_nq^{n-n}p^n}$$

$$= \frac{\sum_{r=0}^{n} r\,{}^nC_rq^{n-r}p^r}{\sum_{r=0}^{n} {}^nC_rq_{n-r}p^r} = \frac{\sum_{r=1}^{n} r.\frac{n}{r}\,{}^{n-1}C_{r-1}q^{n-r}.p.p^{r-1}}{\sum_{r=0}^{n} {}^nC_rq^{n-r}p^r}$$

$$= \frac{np\left(\sum_{r=1}^{n} n-1C_{r-1}p^{r-1}q^{(n-1)(r-1)}\right)}{\sum_{r=0}^{n} {}^nC_rq^{n-r}p^r}$$

$$= \frac{np(q+p)^{n-1}}{(q+p)^n} = np, [\therefore q+p=1]$$

3.(D) $f(x) = \sin^3 x + \lambda \sin^2 x$

$f'(x) = 3 \sin^2 x \cos x + 2\lambda \sin x \cos x = \frac{1}{2} \sin 2x (3 \sin x + 2\lambda)$ **.....(1)**

For maximum or minimum values of f(x), $f'(x) = 0$

$\frac{1}{2} \sin 2x (3 \sin x + 2\lambda) = 0$

$\sin 2x = 0 \qquad \Rightarrow 2x = n\pi$

$\Rightarrow x = \frac{n\pi}{2} \qquad \therefore x = 0 \left(\because -\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

and $3 \sin x + 2\lambda = 0$

$\therefore \sin x = -\frac{2\lambda}{3}$ **.....(2)**

or $-1 < \sin x < 1 \qquad \Rightarrow -1 < -\frac{2\pi}{3} < 1$

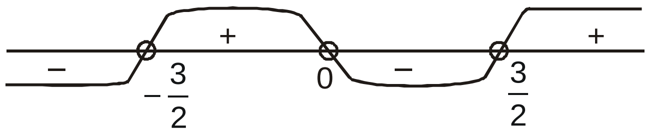
$\Rightarrow -\frac{3}{2} < \lambda < \frac{3}{2}$ **.....(3)**

Also from (1),

$f''(x) = \cos 2x (3 \sin x + 2\lambda) \frac{3}{2} + \sin 2x \cos x$

$$f''\left(\sin^{-1}\left(\frac{-2\lambda}{3}\right)\right) = 0 + 3\left(1 - \frac{4\lambda^2}{9}\right)\left(\frac{-2\lambda}{3}\right)$$

$= \dfrac{-2\lambda(9-4\lambda^2)}{9} = \dfrac{2}{9}\lambda(2\lambda+3)(2\lambda-3)$



If $f''\left(\sin^{-1}\left(\dfrac{-2\lambda}{3}\right)\right) > 0$ (f(x) is minimum)

then $\lambda \quad \left(-\dfrac{3}{2},0\right)\cup\left(\dfrac{3}{2},\infty\right)$ **......(4)**

and If $f''\left(\sin^{-1}\left(-\dfrac{2\lambda}{3}\right)\right) < 0$ (f(x) is maximum)

then $\lambda \quad \left(-\infty,-\dfrac{3}{2}\right)\cup\left(0,\dfrac{3}{2}\right)$ **......(5)**

But from (3), $-\dfrac{3}{2}<\lambda<\dfrac{3}{2}$ **......(6)**

Combining (4), (5), and (6), we get

$\lambda \quad \left(-\dfrac{3}{2},0\right)\cup\left(0,\dfrac{3}{2}\right)$

4.(B) In the given determinant. Replacing R_1 by $R_1 + R_2 + R_3$

$$\begin{vmatrix} a+b+c-x & a+b+c-x & a+b+c-x \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

Taking out (a + b + c – x) as common

$$\Rightarrow (a+b+c-x)\begin{vmatrix} 1 & 1 & 1 \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$$

Replacing C_2 and C_3 by $C_2 - C_1$ and $C_3 - C_1$ respectively

$$\Rightarrow (a+b+c-x)\begin{vmatrix} 1 & 0 & 0 \\ c & b-c-x & a-c \\ b & a-b & c-b-x \end{vmatrix} = 0$$

Expanding with respect to first row.

$$\Rightarrow (a+b+c-x)\begin{vmatrix} b-c-x & a-c \\ a-b & c-b-x \end{vmatrix} = 0,$$

$$\Rightarrow (a+b+c-x)[(b-c-x)(c-b-x)-(a-c)(a-b)] = 0$$

$$\Rightarrow a+b+c-x = 0$$

$$\Rightarrow x = a+b+c$$

$$\Rightarrow x = 0 \text{ (Given that } a+b+c=0)$$

5.(C) Here. given equations are

$x - y + 3z = 4$

$x + z = 2$

$x + y - z = 0$

Write the system of equations in matrix form AX = B.

We get

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$$

Coefficient matrix A = $\begin{bmatrix} 1 & -1 & 3:4 \\ 1 & 0 & 1:2 \\ 1 & 1 & -1:0 \end{bmatrix}$

Rank of A = 2 = $\begin{bmatrix} 1 & -1 & 3:4 \\ 0 & 1 & -2:-2 \\ 0 & 1 & -2:-2 \end{bmatrix}$

By $R_2 = R_2 - R_1$

$R_3 = R_3 - R_2 = \begin{bmatrix} 1 & -1 & 3:4 \\ 0 & 1 & -2:-2 \\ 0 & 1 & 0:-2 \end{bmatrix}$

By $R_3 = R_3 - R_2$

Rank [A : B] = 2

Rank (A) = Rank [A : B]<Number of unknowns

Therefore, the given system of equation has infinite many solutions.

6.(D) Let p denote the probability of getting a head in a single toss and q = 1– p be the probability of getting a tail Consider the variable x = Number of tosses required to obtain the first head. x can take the values 1, 2 , 3 .....n. In general, r tosses will be required if the first r – 1 tosses result in tails only and the last toss gives a head. Since the successive trials are independent the probability of such an occurrence, i.e.

$$P(x = r) = (q.q...q).p = p.q^{r-1}$$

$$\underbrace{(q.q....q)}_{r-1\text{times}}.p = p.q^{r-1}$$

Hence E(x) = $\sum_{r=1}^{\infty}(p.q^{r-1}).r = p\sum_{r=1}^{\infty}(r.q^{r-1})$ = p(1 + 2q + 3q^2 +......) = p.(1 – q)$^{-2}$

Since (1 – x)$^{-2}$ = 1 + 2x + 3x^2 +.... for |x|<1 = $\frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$

7.(C) We have

$$E\left(\frac{X_1+X_2+....+X_n}{X_1+X_2+....+X_n}\right) = E(1) \Rightarrow E\left(\frac{X_1+X_2+....+X_n}{S_n}\right) = 1$$

$$E\left(\frac{X_1}{S_n}\right) + E\left(\frac{X_2}{S_n}\right) + ... + E\left(\frac{X_n}{S_n}\right) = 1$$

Since X_i's, (i = 1, 2, ...,n) are identically distributed random variables, (X_i/S_n), (i = 1, 2,...n) are also identically distributed random variables.

$$n\,E\left(\frac{X_i}{S_n}\right) = 1 \Rightarrow E\left(\frac{X_i}{S_n}\right) = \frac{1}{n};\ i = 1, 2, ..., n \qquad .....(*)$$

Now,

$$E\left(\frac{S_m}{S_n}\right) = E\left(\frac{X_1+X_2+....+X_m}{S_n}\right) = E\left(\frac{X_1}{S_n}+\frac{X_2}{S_n}+.....+\frac{X_m}{S_n}\right)$$

$$= E\left(\frac{X_1}{S_n}\right) + E\left(\frac{X_2}{S_n}\right) + ... + E\left(\frac{X_m}{S_n}\right) = \frac{1}{n}+\frac{1}{n}+....+\frac{1}{n}\ [(\text{m times})][\text{Using}(*)] = \frac{m}{n}, (m<n)$$

8.(D) Let X be a r.u. with probability function :

$$P(X = \pm 2^x) = \begin{cases} \frac{e^{-1}}{2x!}; x = 0,1,2,.... \\ 0, \text{otherwise} \end{cases}$$

Since the distribution is symmetric about the line X = 0, all moments of odd order about origin vanish, i.e.

$$E(X^{2r+1}) = 0 \ \mu_{2r} + 1 = 0.$$

$$E(X^{2r}) = \sum_{x=0}^{\infty}\left(\pm 2^x\right)\left(\frac{1}{2ex!}\right)^{2r} = \frac{1}{e}\sum_{x=0}^{\infty}\frac{(2^{2r})^x}{x!} = \frac{1}{e}\exp(2^{2r}) = \exp(2^{2r}-1)$$

Thus all the moments of X exist. The m.g.f of X, if it exists, is given by

$$M_X(t) = \sum_{x=0}^{\infty}\left\{(e^{t.2^x} + e^{-t.2x})\frac{1}{2e\,x!}\right\} = e^{-1}\sum_{x=0}^{\infty}\left\{\frac{\cosh(t2^x)}{x!}\right\},$$

Which converges only for t = 0.

As an illustration of continuous probability distribution consider Pareto distribution with p.d.f:

$$p\,(x) = \frac{\theta a^{\theta}}{x^{\theta+1}}; x \geq a, \theta > 1$$

$$E(X^r) = \theta\ a^{\theta}\int_a^{\infty} x^{r-\theta-1}dx = \theta a^{\theta}\left|\frac{x^{r-\theta}}{r-\theta}\right|_a^{\infty}$$, which is finite iff $r - \theta < 0 \Rightarrow \theta > r$

and then $$E(X^r) = \theta\ a^{\theta}\left[0 - \frac{a^{r-\theta}}{r-\theta}\right] = \frac{\theta . a^r}{\theta - r}; \theta > r$$

However, the m.g.f. is given by :

$$M_X\,(t) = \theta a^{\theta} a^{\theta}\int_a^{\infty}\frac{e^{tx}}{x^{\theta+1}}dx,$$

Which does not exist, since e^{tx} dominates $x^{\theta+1}$ and $(e^{tx}/x^{\theta+1}) \to \infty$ as $x \to \infty$ and hence the integral is not convergent.

9.(B) We shall assume that X and Y are continuous random variables.

$$P(AB) = P[(x : a < x < b)(y : c < y < d)] = \int_a^b\int_c^d f_{xy}(x,y)\,dxdy = \int_a^b\int_c^d g_x(x)h_y(y)\,dxdy,$$

since A and B are independent $= \left[\int_c^d g_X(x)dx\right]\left[\int_c^d h_Y(y)dy\right] = P(A)P(B)$

which proves the result.

10.(B) Marginal P.d.f of X is given by :

$$f_X(x) = \int_0^1 f(x,y)dy = \int_0^1 (x+y)dy = \left[xy + \frac{y^2}{2}\right]_0^1 = \left(\frac{x+1}{2}\right),\ 0 \le x \le 1$$

then the condition distribution of y for X = x is given by :

$$f_{xy}(Y = y|X = x) = \frac{f_{xy}(x,y)}{f_x(x)} = \frac{x+y}{\frac{1}{2}(2x+1)} = \frac{2(x+y)}{2x+1}$$

$$f_{xy}(Y = y|X = x) = \frac{2(x+y)}{2x+1};\ 0 \le y \le 1.$$

11.(A) L.H.L $= \lim_{x\to\pi/4-} f(x) = \lim_{h\to 0} a\left(\frac{\pi}{4} - h\right) + b = a\left(\frac{\pi}{4}\right) + b$

R.H.L $= \lim_{x\to\pi/4+} f(x) = \lim_{h\to 0} \cos\left(\frac{\pi}{4} + h\right) = \frac{1}{\sqrt{2}}$

For continuity L.H.L = R.H.L

$$\Rightarrow a\left(\frac{\pi}{4}\right)+b=\frac{1}{\sqrt{2}} \qquad \textbf{....(1)}$$

$$\text{L.f.}\left(\frac{\pi}{4}\right) = \lim_{h\to 0}\frac{f\left(\frac{\pi}{4}-h\right)-f\left(\frac{\pi}{4}\right)}{-h} = \lim_{h\to 0}\frac{\left\{a\left(\frac{\pi}{4}-h\right)+b\right\}-\left\{a\left(\frac{\pi}{4}\right)+b\right\}}{-h} = a$$

$$\text{R.f}\left(\frac{\pi}{4}\right)=\lim_{h\to 0}\frac{f\left(\frac{\pi}{4}+h\right)-f\left(\frac{\pi}{4}\right)}{h}=\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{4}+h\right)-\cos\frac{\pi}{4}}{h}=\lim_{h\to 0}\frac{-2\sin\left(\frac{\pi}{4}+\frac{h}{2}\right).\sin h}{h}$$

$$=-2\sin\frac{\pi}{4}.\lim_{h\to 0}\frac{\sin h}{h} =-2\times\frac{1}{\sqrt{2}}\times 1=\frac{-1}{\sqrt{2}}$$

$$\therefore \text{L.f}\left(\frac{\pi}{4}\right)=\text{R f}\left(\frac{\pi}{4}\right)$$

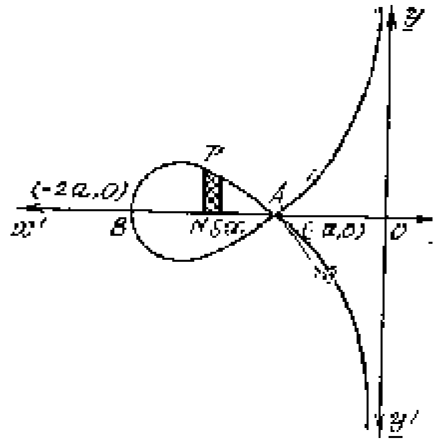
$$\Rightarrow a=-\frac{1}{\sqrt{2}}$$

Hence, from **(1)**

$$b=\frac{1}{\sqrt{2}}\left(\frac{\pi}{4}+1\right)$$

12.(D) The curve $x^2+y^2=a^2$ is a circle with centre 0 (0, 0) and radius a.

The equation x + y = a or (x/a) + (y/a) = 1 represents a straight line which cuts off intercepts a and a from positive directions of x and y axis.



Hence the points of intersection of $x^2 + y^2 = a^2$ and x + y = a are A (a, 0) and B (0, a) respectively.

So the area under consideration is the area AQPBP' Q' A.

$\therefore$ The required area $= \int_{x=0}^{a} \int_{y=P'N}^{PN} dx\, dy$, where $PN = \sqrt{(a^2 - x^2)}$ and P'N = a – x

$$= \int_{x=0}^{a} \int_{y=(a-x)}^{\sqrt{(a^2-x^2)}} dx\, dy = \int_0^a [y]_{(a-x)}^{\sqrt{(a^2-x^2)}} dx = \int_0^a \left[\sqrt{(a^2 - x^2)} - (a - x)\right] dx$$

$$= \left[\left\{\tfrac{1}{2} x\sqrt{(a^2 - x^2)} + \tfrac{1}{2} a^2 \sin^{-1}(x/a)\right\} - ax + \tfrac{1}{2} x^2\right]_0^a$$

$$= \left[\tfrac{1}{2} a^2 \left(\tfrac{1}{2}\pi\right) - a^2 + \tfrac{1}{2} a^2\right] = \tfrac{1}{2} a^2 \left(\tfrac{1}{2}\pi - 1\right) = \tfrac{1}{4} a^2 (\pi - 2).$$

13.(D) Here first of all we shall determine the constant y_0 from the consideration that the area under a probability curve is unity.

$$\therefore \quad y_0 \int_\alpha^\infty \exp[-\beta(x-\alpha)dx \Rightarrow y_0 \left|\frac{e^{-\beta(x-\alpha)}}{-\beta}\right|_\alpha^\infty = 1 \Rightarrow -\frac{y_0}{\beta}(0-1) = 1 \Rightarrow y_0 = \beta$$

$\therefore \qquad f(x;\alpha,\beta) - \beta e^{-\beta(x-\alpha)}, \alpha \le x \le \infty$

If $x_1, x_2, x_3 \ldots, x_n$ is a random sample of n observation from this population, then

$$L = \prod_{i=1}^{n} f(x_i;\alpha,\beta) = \beta^n \exp\left\{-\beta \sum_{i=1}^{n} (x_i - \alpha)\right\} = \beta^n \exp\left[-n\beta(\bar{x} - \alpha)\right]$$

$\therefore \qquad \log L = n \log \beta - n\beta\ (\bar{x} - \alpha)$ **.......(i)**

The likelihood equations for estimating α and β give

$$\frac{\partial}{\partial \alpha} \log L = 0 = n\beta \qquad \textbf{.......(ii)}$$

and $$\frac{\partial}{\partial \alpha} \log L = 0 = \frac{n}{\beta} - n(\bar{x} - \alpha) \qquad \textbf{.......(iii)}$$

Equation (2) given $\beta = 0$, which is obviously inadmissible and this on substitution in (3) gives $\alpha = \infty$. Thus the likelihood equations fail to give us valid estimates of α and β and we try to locate M.L.Es for α and β by maximising L directly.

L is maximum $\Rightarrow$ log L maximum

From (1), log L is maximum (for any value of β), if $(\bar{x} - \alpha)$ is minimum, which is so if α is maximum and

If $x_{(1)}, x_{(2)} \ldots\ldots x_{(n)}$ is ordered sample from this population then $\alpha \le x_{(1)} \le x_{(2)} \ldots\ldots \le x_{(n)}$ so that the maximum value of α consistent with the sample is $x_{(1)}$, the smallest sample observation, i.e., $\hat{\alpha} = x_{(1)}$.

Consequently, (3) gives $\frac{1}{\beta} = \bar{x} - \hat{\alpha} = \bar{x} - x_{(1)} \Rightarrow \hat{\beta} = \frac{1}{\bar{x} - x_{(1)}}$

Hence M.L.E's for α and β are given by : $\hat{\alpha} = x_{(1)}$ and $\hat{\beta} = \frac{1}{\bar{x} - x_{(1)}}$.

14.(A) $f(x,\theta) = \frac{x}{1+\theta}$; $0 \le x \le 1$

$$\mu_1' = E(x) = \int_0^1 x \cdot \frac{x}{1+\theta} dx = \frac{1}{1+\theta} \int_0^1 x^2\, dx = \frac{1}{1+\theta}\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{1+\theta}\left[\frac{1}{3}\right] = \frac{1}{3(1+\theta)}$$

$$\Rightarrow\ 1 + \theta = \frac{1}{3\mu_1'} \quad \Rightarrow \quad \theta = \frac{1}{3\mu_1'} - 1$$

Replace μ_1' by M_1'.

$$\therefore \qquad \theta = \frac{1}{3M_1'} - 1.$$

15.(A) $f(x) = x^2 e^{-x}$

$f'(x) = 2x.e^{-x} - x^2 e^{-x}$

$f'(x) = 0$ (for maxima and minima)

$xe^{-x}(2 - x) = 0$

$x = 0, x = 2$

$f''(x) = 2e^{-x} - 2xe^{-x} - 2x.e^{-x} + x^2 e^{-x} = e^{-x}[2 - 2x - 2x + x^2] = e^{-x}[x^2 - 4x + 2]$

$f''(0) > 0$ $x = 0$ point of minima

$f''(2) < 0$ $x = 2$ point of maxima

$f(0) = 0$

$f(2)s = 4e^{-2} = \frac{4}{e^2}$

16.(A) The given equation can be rewritten as

$$-(x+a)\frac{dy}{dx} = ay^2 \qquad \text{or} \qquad \frac{(x+a)}{y^2}\frac{dy}{dx}+\frac{1}{y} = 0 \quad ...(1)$$

Putting $\frac{1}{y} = v$ or $-\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx}$ in (1) we get

$$(x+a)\frac{dv}{dx}+v = a \qquad \text{or} \qquad \frac{dv}{dx}+\frac{v}{x+a} = \frac{a}{x+a}, \quad ...(2)$$

which is a linear equation in v.

Its integrating factor = $e^{\int[1/(x+a)]dx} = e^{\log(x+a)} = x + a$

Multiplying both sides of (2) by this integrating factor and integrating

we have $v(x+a) = C + \int \frac{a}{x+a}.(x+a)dx = C + a\int dx,$

where C is an arbitrary constant.

or $v(x + a) = C + ax$ or $(x + a)/y = C + ax$, $v = 1/y$

or $x + a = y(C + ax)$

Put $x = 0$, $y = 1$ we get $\boxed{c = a}$

$\frac{x+a}{a} = y(1+x)$

17.(B) The C-coordinate vectors of the image of $\mathbf{b_1}$ and $\mathbf{b_2}$ are

$$[T(\mathbf{b_1})]_C = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{b_2})]_C = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

Hence

$$M = \begin{bmatrix} 3 & 4 \\ -2 & 7 \\ 5 & -1 \end{bmatrix}$$

If B and C are bases for the same space V and if T is the identity transformation T(x) = x for x in V, then the matrix M in (4) is just a change-of-coordinates matrix.

18.(C) X ~ N (0, 1), $\therefore$ E(x) = 0, = Var x = 1, $Ex^2 = 1$, $Ex^3 = 0$, $Ex^4 = 3$.

$E(Y) = E(a + bx + cx^2) = a + b.0 + c.1 = a + b.0 + c.1 = a + c$, $\sigma_y{}^2 = E[Y - (a + c)]^2$

$= E(Y^2) - (a + c)^2. = b^2 + 2c^2 - ac$

$\therefore r(X, Y) = \text{Cov}(X, Y)/\sigma_X \sigma_y = \dfrac{b}{\sqrt{b^2 + 2c^2 - ac}}$.

19.(A) Given that X_1 and X_2 have the joint pdf

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 10x_1x_2^2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

where $Y_1 = X_1 / X_2$ and $Y_2 = X_2$. Hence, the inverse transformation is $x_1 = y_1y_2$ and $x_2 = y_2$ which has the Jacobian

$$J = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = y_2.$$

The inequalities defining the support S of (X_1, X_2) becomes

$$0 < y_1y_2,\ y_1y_2 < y_2, \text{ and } y_2 < 1.$$

These inequalities are equivalent to

$$0 < y_1y_2,\ y_1y_2 < y_2, \text{ and } y_2 < 1.$$

These inequalities are equivalent to

$$0 < y_1 < 1 \text{ and } 0 < y_2 < 1,$$

which defines the support set T of (Y_1, Y_2). Hence, the pdf of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1, y_2) = 10y_1\, y_2\, y_2^2\, |\, y_2\, | = 10y_1\, y_2^4, \qquad (y_1, y_2) \in T.$$

The marginal pdf are :

$$f_{Y_1}(y_1) = \int_0^1 10y_1 y_2^4 dy_2 = 2y_1, \qquad 0 < y_1 < 1$$

zero elsewhere.

20.(C) Let X_1 and X_2 have the joint pdf.

$$f(x_1, x_2) = \begin{cases} 6x_2 & 0 < x_2 < x_1 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then the marginal pdf of X_1 is

$$f_1(x_1) = \int_0^{x_1} 6x_2 dx_2 = 3x_1^2, \qquad 0 < x_1 < 1,$$

zero elsewhere. The conditional pdf of X_2, given $X_1 = x_1$, is

$$f_{2|1}(x_2 \mid x_1) = \frac{6x_2}{3x_1^2} = \frac{2x_2}{x_1^2}, \qquad 0 , x_2 < x_1,$$

zero elsewhere, where $0 < x_1 < 1$. The conditional mean of X_2, given $X_1 = 1/2$, is

$$E\left(X_2 \mid x_1 = \frac{1}{2}\right) = \int_0^{x_1} x_2 \left(\frac{2x_2}{x_1^2}\right) dx_2 = \frac{2}{3} x_1, \qquad 0 < x_1 < 1.$$

i.e. $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$

21.(A) The pmf of X on its support is

$$f(x, \theta) = e^{-\theta}\frac{\theta^x}{x!} = \exp\{\log(\theta)x + \log(1/x!) + (-\theta)\}.$$

Hence, the Poisson distribution is a member of the regular exponential class, with $p(\theta) = \log(\theta)$, $q(\theta) = -\theta$, and $K(x) = x$. Therefore if $X_1, X_2, \ldots, X_n$ denotes a random sample on X then the statistic $Y_1 = \sum_{i=1}^{n} X_i$ is sufficient.

22.(A) Null Hypothesis, (H_0) : The sample has been drawn from the population with mean $\mu = 3.25$ cms. and S.D. $\sigma = 2.61$ cms.

Alternative Hypothesis, $H_1 : \mu \neq 3.25$ (Tw-tailed).

Test Statistic. Under H_0, the test statistic is : $Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0, 1)$, (since n is large.)

Here, we are given : $\bar{x} = 3.4$cms., n = 900cms., $\mu = 3.25$ cms. and $\sigma = 2.61$ cms.

$$\therefore \qquad Z = \frac{3.40-3.25}{2.61/\sqrt{900}} = \frac{0.15\times 30}{2.61} = 1.73$$

Since | Z | < 1.96, we conclude that the data don't provide us any evidence against the null hypothesis (H_0) which may, therefore, be accepted at 5% level of significance.

95% fiducial limits for the population mean μ are :

$$\bar{x} \pm 1.96\left(\sigma/\sqrt{n}\right) = 3.40 \pm 1.96\left(2.61/\sqrt{900}\right) = 3.40 \pm 0.1705, \text{ i.e., } 3.5705 \text{ and } 3.2295$$

98% fidcial limits for μ are given by :

$$\bar{x} \pm 2.33\frac{\sigma}{\sqrt{n}} = 3.40 \pm 2.33 \times \frac{2.61}{30} = 3.40 \pm 0.2027, \text{ i.e., } 3.6027 \text{ and } 3.1973$$

Remark. 2.33 is the value z_1 of Z from standard normal probability integrals, such that

$$P(|Z| > z_1) = 0.98 \Rightarrow P(Z > z_1) = 0.49$$

23.(A) $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\int_2^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}} = \lim\limits_{x\to\frac{\pi}{4}} \dfrac{f(\sec^2 x).\frac{d}{dx}(\sec^2 x) - f(2).\frac{d}{dx}(2)}{2x} = \lim\limits_{x\to\frac{\pi}{4}} \dfrac{f(\sec^2 x).2\sec x.\sec x\tan x - 0\tan x}{2x}$

$= \dfrac{f(2).2.2.1}{2.\frac{\pi}{4}} = \dfrac{8}{\pi} f(2)$ $\quad\quad\quad\quad$ $\therefore$ (A) holds.

24.(A) The total number of elementary events associated to the random experiment of throwing three dice is $6 \times 6 \times 6 = 6^3$.

Favourable number of elementary events

= Coefficient. of x^k in $(x + x^2 + x^3 + \ldots + x^6)^3$ = Coefficient of x^{k-3} in $\left(\dfrac{1-x^6}{1-x}\right)^3$

= Coefficient of x^{k-3} in $(1 - x^6)^3 (1 - x)^{-3}$ = Coefficient of x^{k-3} in $(1 - x)^{-3}$

$[\because 0 \le k - 3 \le 5]$

$= {}^{k-3+3-1}C_{3-1}$ [coefficient of x^n in $(1 - x)^{-r} = {}^{n+r-1}C_{r-1}$] $= {}^{k-1}C_2$

$= \dfrac{(k-1)(k-2)}{2}$

Hence, the probability of the required event $= \dfrac{(k-1)(k-2)}{2\times 6^3} = \dfrac{(k-1)(k-2)}{432}$.

25.(A) Given that X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} 2 & 0 < x_1 < x_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then the marginal probability density functions are, respectively,

$$f_1(x_1) = \begin{cases} \int_{x_1}^{1} 2dx_2 = 2(1 - x_1) & 0 < x_1 < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

and $$f_2(x_2) = \begin{cases} \int_{0}^{x_2} 2dx_1 = 2x_2 & 0 < x_2 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

The conditional pdf of X_1, given $X_2 = x_2$, $0 < x_2 < 1$, is

$$f_{1|2}(x_1 \mid x_2) = \begin{cases} \frac{2}{2x_2} = \frac{1}{x^2} & 0 < x_1 < x_2 \\ 0 & \text{elsewhere.} \end{cases}$$

Here the conditional mean and the conditional variance of X_1, given $X_2 = x_2$, are respectively,

$$E(X_1|x_2) = \int_{-\infty}^{\infty} x_1 f_{1|2}(x_1 \mid x_2)\,dx_1 = \int_{-\infty}^{\infty} x_1 f_{1|2}(x_1 \mid x_2)\,dx_1 = \frac{x_2}{2},\ 0 < x_2 < 1,$$

and $$\text{var}(X_1 \mid x_2) = \int_0^{x_2} \left(x_1 - \frac{x_2}{2}\right)^2 \left(\frac{1}{x_2}\right) dx_1 = \frac{x_2^2}{12},\quad 0 < x_2 < 1.$$

26.(A) $(2x \cos y + y^2 \cos x)dx + (2y \sin x - x^2 \sin y)\,dy = 0$

$\Rightarrow \quad d(x^2 \cos y) + d(y^2 \sin x) = 0$

On integrating, $x^2 \cos y + y^2 \sin x = c$

27.(C) The total number of elementary events associated to the random experiment of throwing three dice is $6 \times 6 \times 6 = 6^3$.

Favourable number of elementary events

$= \text{Coefficient. of } x^k \text{ in } (x + x^2 + x^3 + \ldots + x^6)^3 = \text{Coefficient of } x^{k-3} \text{ in} \left(\frac{1-x^6}{1-x}\right)^3$

$= \text{Coefficient of } x^{k-3} \text{ in } (1 - x^6)^3 (1 - x)^{-3}$

$= \text{Coefficient of } x^{k-3} \text{ in } (1 - x)^{-3}$ $\quad [\because 0 \le k - 3 \le 5]$

$= {}^{k-3+3-1}C_{3-1}$ [coefficient of x^n in $(1 - x)^{-r} = {}^{n+r-1}C_{r-1}$]

$= {}^{k-1}C_2 = \frac{(k-1)(k-2)}{2}$

Hence, the probability of the required event $= \frac{(k-1)(k-2)}{2\times 6^3} = \frac{(k-1)(k-2)}{432}$.

28.(D) Given that $X \sim G\left(5, \frac{1}{3}\right)$

where $f(x \mid \alpha, \lambda) = \begin{cases} \frac{\lambda^\alpha}{\sqrt{\alpha}} x^{\alpha-1} e^{-\lambda x} & , x > 0,\ \lambda > 0,\ \alpha > 0 \\ 0 & , \text{otherwise} \end{cases}$

Now $\quad P(X > 6) = \int_6^\infty \frac{\left(\frac{1}{3}\right)^5}{\sqrt{5}} x^4 e^{-\frac{x}{3}} dx = \frac{1}{8(3)^6} \int_0^\infty x^4 e^{-x/3} dx$

$= \frac{1}{8(3)^6} \left[\frac{x^4 e^{-x/3}}{-1/3} + \frac{4}{1/3} \int x^3 e^{-x/3}\, dx \right]_6^\infty = \frac{12}{8(3)^6} \left[0 + \frac{x^3 e^{-x/3}}{-1/3} + \frac{3}{1/3} \int x^2 e^{-x/3} dx \right]_6^\infty$

$= \frac{3^2.3.4}{8(3)^6} \left[0 + x^2 e^{-x/3} + \frac{2}{1/3} \int x e^{-x/3} dx \right]_6^\infty = \frac{3^4.8}{8(3)^6} \left[0 + \frac{x e^{-x/3}}{-1/3} - 3^2 e^{-x/3} \right]_6^\infty = \frac{3^5.8}{3^6.8} \left[0 - e^{-\infty} + e^{-2} \right] = \frac{e^{-2}}{3}$

29.(B) We know that taylor's expansion of f(x) about origin is

$f(0 + h) = f(0) + hf'(0) \; + \frac{h^2}{2!} f''(0) + \frac{h^3}{3!} f'''(0) + ...$

or $\quad f(0 + x) = f(0) + xf'(0) \; + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ...$

or $\quad f(0 + x) = f(0) + xf'(0) \; + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f''(0) + ...$

$\because f(0) \; 1$

$f'(0) = 1 + [f(0)]^3 = 1 + [1]^3 = 2$

$f''(0) = 3[f(0)]^2 \times f'(0) = 3[1]^2 \times (2) = 6$

$f''(0) = 3[f''(0) f((0))^2] + 6f(0) [f'(0)]^2 = 3[6(1)^2] + 6[1(2)^2] = 18 + 24 = 42$

then coefficient of f'''(0) in taylor's expansion is given as

$\frac{42}{3!} = \frac{42}{6} = 7$

30.(C) Given $\lim_{x\to\infty} \frac{\int_0^{2x} te^{t^2} dt}{e^{4x^2}} \left(\frac{\infty}{\infty} \text{ form}\right)$

then $\lim_{x\to\infty} \frac{\int_0^{2x} te^{t^2} dt}{e^{4x^2}} = \lim_{x\to\infty} \frac{\frac{d}{dx}\int_0^{2x} te^{t^2} dt}{\frac{d}{dx} e^{4x^2}} = \lim_{x\to\infty} \frac{\left[\int_0^{2x} \frac{\partial}{\partial x} t e^{t^2} dt + 2xe^{(2x)^2}(2) - 0\right]}{e^{4x^2}(8x)} = \lim_{x\to\infty} \frac{\left[0 + 4xe^{4x^2} - 0\right]}{8xe^{4x^2}}$

$= \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$

31.(A,B,D) Since $f_{xy}(0, 0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$

But $\quad f_y(0, 0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0$

$$f_y(h,0) = \lim_{k\to 0}\frac{f(h,k)-f(h,0)}{k} = \lim_{k\to 0}\frac{hk(h^2-k^2)}{k\cdot(h^2+k^2)} = h$$

So, $f_{xy}(0,0) = \lim_{h\to 0}\frac{h-0}{h} = 1$

again $f_{yx}(0,0) = \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}$

but $f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \frac{0}{h} = 0$

$$f_x(0,k) = \lim_{h\to 0}\frac{f(h,k)-f(0,k)}{h} = \lim_{h\to 0}\frac{hk(h^2-k^2)}{h(h^2+k^2)} = -k$$

So, $f_{yx}(0,0) = \lim_{k\to 0}\frac{-k-0}{+k} = -1$

Since $f_{xy}(0,0) = 1$ and $f_{yx}(0,0) = -1$

Hence $f_{xy}(0,0) \neq f_{yx}(0,0)$

32.(A,C) Given differential equation is

$$(a^2 - 2xy - y^2)dx - (x+y)^2dy = 0$$

$$\Rightarrow a^2dx - 2xydx - y^2dx - x^2dy - 2xydy - y^2dy = 0$$

$$\Rightarrow a^2dx - y^2dy - d(x^2y) - d(xy^2) = 0$$

Now integrated both side, we have

$$a^2x - \frac{y^3}{3} - x^2y - xy^2 = c$$

but given that $y(0) = 1$ then

$0 - \frac{1}{3} - 0 - 0 = c$ i..e $\boxed{c = -\frac{1}{3}}$

then the required solution is

$$a^2x - x^2y - xy^2 \qquad -\frac{y^3}{3} = -\frac{1}{3}$$

$$3a^2x - 3x^2y - 3y^2x - y^3 + 1 = 0$$

33.(B,C) We know that square of standard normal variate becomes χ^2-variate with and sum of two χ^2-variate is again χ^2- variate with 2-degree of freedom if both variate have save degree of freedom 1.

i.e. $X_1^2 + X_2^2 \sim \chi^2_{(2)}$

So, $P\left(X_1^2 + X_2^2 \le 1\right) = P\left(\chi^2_{(2)} < 1\right)$

$$P\left(\chi^2_{(2)} \le 1\right) = \int_0^1 \frac{1}{2^{2/2}\sqrt{2/2}} e^{-x/2} x^{\frac{2}{2}-1} dx = \int_0^1 \frac{1}{2} e^{-x/2} dx = \left[\frac{1}{2}\frac{e^{-x/2}}{-1/2}\right]_0^1 = \left[-e^{-1/2} + e^{-0}\right]$$

$$= 1 - \frac{1}{\sqrt{e}} \text{ or } \frac{\sqrt{e}-1}{\sqrt{e}} = \frac{e-\sqrt{e}}{e}$$

34.(A,C,D) The critical region is given as

$$W = \{x : \bar{x} \ge K\}$$

where $\alpha = P\{x \in W / H = H_0\} = P\{\bar{x} \in k / \mu = 10\}$ ... (1)

Since $X \sim N(\mu, 1) \Rightarrow \bar{X} \sim N\left(\mu, \frac{1}{10}\right)$ $[\text{as } n = 10]$

Under H_0 $\bar{X} \sim N\left(8, \frac{1}{10}\right) \Rightarrow Z = \frac{\bar{X} - E(\bar{X})}{\sigma_{\bar{x}}} = \frac{\bar{X} - 8}{1/\sqrt{10}} = \sqrt{10}\left(\bar{X} - 8\right)$

again under H_0, $\bar{X} = K$ So, $Z = \sqrt{10}\,(k - 8)$

Now $P\left\{Z > \sqrt{10}(k-8)\right\} = 0.05$

$\Rightarrow \sqrt{10}(k-8) = 1.65$ [By the normal table]

$\Rightarrow K = 8 + 0.52$

$\Rightarrow K = 8.52$

35.(A,B,C) $L = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n (x_1 x_2 \ldots x_n)^{\theta-1}$,

So $\frac{L_{H_1}}{L_{H_0}} = \frac{(-2)^n (x_1 x_2 \ldots x_n)1}{(1)^n (x_1 x_2 \ldots x_n)^0}$, $= 2^n (x_1 x_2 \ldots x_n) \geq k = (x_1 x_2 \ldots x_n)^{1/n} \geq \frac{k^{1/n}}{2}$

$= (x_1 x_2 \ldots x_n)^{1/n} \geq k_1$ $\left[\text{where } k_1 = \frac{k^{1/n}}{2}\right]$

36.(A,D) Since $\frac{E}{F}$ and $\frac{\bar{E}}{F}$ are complementary events.

Therefore, $P\left(\frac{E}{F}\right) + P\left(\frac{\bar{E}}{F}\right)$

$\Rightarrow \frac{P(E \cap F)}{P(F)} + \frac{P(\bar{E} \cap F)}{P(F)} = \frac{P(E \cap F) + P(F) - P(E \cap F)}{P(F)} = 1.$

$1 - P\left(\frac{\bar{E}}{F}\right) - P\left(\frac{E}{F}\right) = 0$

37.(A,B) E_1 : coin is fair, E_2 : coin is baised, a second toss shows tail.

$P(E_1/A) = \frac{P(A/E_1)P(E_1)}{P(A/E_1)P(E_1) + P(A/E_2)P(E_2)} = \frac{\frac{m}{N}.\frac{1}{2}.\frac{1}{2}}{\frac{m}{N}.\frac{1}{2}.\frac{1}{2} + \frac{N-m}{N}.\frac{2}{3}.\frac{1}{3}} = \frac{9m}{9m + 8N - 8m}$

$$= \frac{9m}{8N+m} = \frac{18m^2}{16Nm+2m^2}$$

38.(A,B,D) We know that if $x \sim G(\lambda)$ distribution then m.g.f. of X is given as

$$M_x(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x)\,dx = \frac{1}{\sqrt{\lambda}}\int_0^{\infty} e^{tx}.e^{-x}x^{\lambda-1}dx = \frac{1}{\sqrt{\lambda}}\int_0^{\infty} e^{-(1-t)x}x^{\lambda-1}dx$$

$$= \frac{\sqrt{\lambda}}{(1-t)^{\lambda}}\cdot\frac{1}{\sqrt{\lambda}}, \quad |t|<1 \quad = \frac{1}{(1-t)^{\lambda}}$$

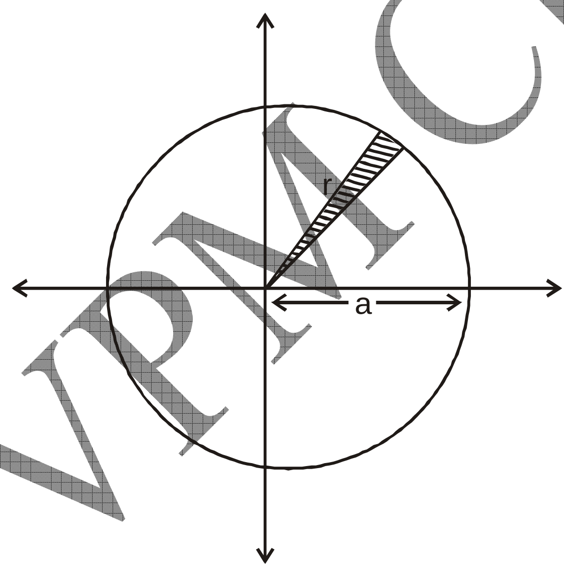
with comparison of our result we can say that

$\lambda = 5$ So X ~ G(S)

39. (A,B,D) Putting x = r cosθ, y = r sinθ and dx dy = rdrdθ in given integral

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dx\,dy = \int_0^a \int_0^{\frac{\pi}{2}} r^2\sin^2\theta\cdot r\cdot r\,dr\,d\theta$$

Here limits are find form the figure



$$= \int_0^a r^4 \left[\int_0^{\frac{\pi}{2}} \frac{1-\cos 2\theta}{2} d\theta\right] dr = \int_0^a r^4 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} dr = \int_0^a r^4 \left[\frac{\pi}{4} - 0\right] dr = \frac{\pi}{4}\left[\frac{r^5}{5}\right]_0^9 = \frac{\pi a^5}{20}$$

40. (B,D) Given differential equation can be written as

$$(1 - x^2)\frac{dy}{dx} + xy = xy^2(1-x^2)$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{x}{1-x^2}\frac{y}{y^2} = x$$

let $\frac{1}{y} = t \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

So, $-\frac{dt}{dx} + \frac{x}{1-x^2}t = x$

I.F. $e^{-\int \frac{x}{1-x^2}dx}$ let $1 - x^2 = v$ $\Rightarrow -2xdx = dv$ or $-xdx = \frac{+dv}{2}$

$\therefore$ I.F. $= e^{\int \frac{dv}{2v}} = e^{\frac{1}{2}\log v} = \sqrt{1-x^2}$

then the general solution of equation is

$$t \cdot \sqrt{1-x^2} = \int -x\sqrt{1-x^2}\, dx + c$$

again let $1 - x^2 = w$

$\Rightarrow -xdx = dw/2$

$$\Rightarrow \frac{1}{y}\sqrt{1-x^2} = \frac{1}{2}\int \sqrt{\omega}\, dw + c$$

$$\Rightarrow \frac{1}{y}\sqrt{1-x^2} = \frac{1}{2}\left[\frac{2}{3}\omega^{3/2}\right] + c$$

$$\Rightarrow \frac{1}{y}\sqrt{1-x^2} = \frac{1}{3}(1-x^2)^{3/2} + c$$

$$\Rightarrow \qquad \sqrt{1-x^2} = y\left[c + \frac{1}{3}(1-x^2)^{3/2}\right]$$

using initial condition y(0) = 1 we have

$$1 = 1\left[c + \frac{1}{3}\right]$$

i.e. $\boxed{c = \frac{2}{3}}$

So, general solution is $\sqrt{1-x^2} = y\left[\frac{2}{3} + \frac{1}{3}(1-x^2)^{3/2}\right]$ = $\frac{3}{2}\sqrt{1-x^2} = y\left[1 + \frac{(1-x^2)^{3/2}}{2}\right]$ **Ans.**

41. 4.06

The unbiased estimators of the population mean (μ) and the population variance (σ^2) are

$$\bar{x} = \frac{\sum x_i}{n} \qquad \text{and } S^2 = \frac{\sum (x_i - \bar{x})^2}{(n-1)} \text{ resp.}$$

Here n = 5 and $\bar{x} = \frac{\sum x_i}{n} = \frac{95}{5}$ = 19

and, $\sum (x_i - \bar{x})^2 = (-5)^2 + 0^2 + (-2)^2 + 1^2 + 6^2 =$ 25 + 4 + 1 + 36 = 66

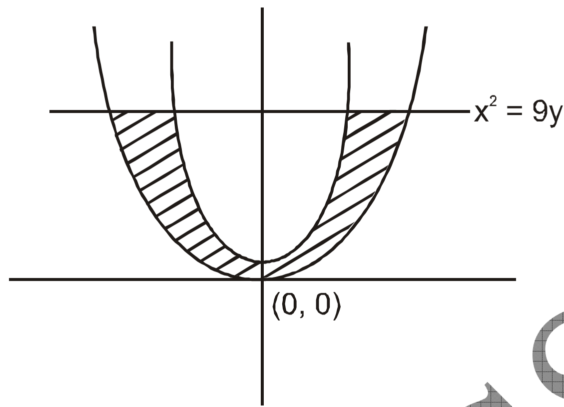
Hence, $s^2 = \frac{66}{4} = 16.5$

$\Rightarrow$ s = $\sqrt{16.5}$ = 4.06

Hence the estimators of σ is 4.06.

42. 9.426

Both curves are symmetric about y-axis. It is advisable to integrate about y-axis



$x = \pm\frac{\sqrt{y}}{2}$ $x = \pm 3\sqrt{y}$

$$2\int_0^2 \left(3\sqrt{y} - \frac{1}{2}\sqrt{y}\right) dy = 5\int_0^2 \sqrt{y}\, dy = \frac{20\sqrt{2}}{3} = 9.426$$

43. **7**

since A has 9 columns, (rank A) + 2 = 9, and hence rank A = 7.

44. **2**

Since $X \sim N(\mu, 81)$ and $Y \sim N(\mu, 144)$

then $X + 2Y \sim N(3\mu, 657)$ and $2X + Y \sim N(\mu, 387)$

Now $P(X + 2Y \leq 3) = P(2X - Y \geq 4)$

$\Rightarrow$ $P\left(Z \leq \frac{3-3\mu}{\sqrt{657}}\right) = P\left(Z \leq \frac{4-\mu}{\sqrt{387}}\right)$

[using $P(z \leq a) = P(z \geq b) \Rightarrow a = -b$]

then $-\left(\frac{3-3\mu}{\sqrt{657}}\right) = \left(\frac{4-\mu}{\sqrt{387}}\right)$

$\Rightarrow \qquad \frac{3\mu - 3}{\sqrt{657}} = \frac{4-\mu}{\sqrt{387}}$

$\Rightarrow \qquad \frac{3\mu - 3}{25.63} = \frac{4-\mu}{19.67}$

$\Rightarrow \qquad 59\mu - 59 = 102.52 - 25.63\mu$

$\Rightarrow \qquad 84.63\,\mu = 161.52$

$\Rightarrow \qquad \mu = 1.908$

2 (approximate)

45. **0.375**

We have, for instance,

$$P\left(0 < x_1 < \frac{3}{4}, \frac{1}{3} < x_2 < 2\right) = \int_{1/3}^{2}\int_{0}^{3/4} f(x_1, x_2)\,dx_1\,dx_2 = \int_{1/3}^{1}\int_{0}^{3/4} 6x_1^2 x_2\,dx_1\,dx_2 + \int_{1}^{2}\int_{0}^{3/4} 0\,dx_1\,dx_2$$

$$= \frac{3}{8} + 0 = \frac{3}{8} = 0.375$$

46. **0.33**

Let X_1 and X_2 have the joint pdf.

$$f(x_1, x_2) = \begin{cases} 6x_2 & 0 < x_2 < x_1 < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then the marginal pdf of X_1 is

$$f_1(x_1) = \int_0^{x_1} 6x_2\,dx_2 = 3x_1^2, \qquad 0 < x_1 < 1,$$

zero elsewhere. The conditional pdf of X_2, given $X_1 = x_1$, is

$$f_{2|1}(x_2 \mid x_1) = \frac{6x_2}{3x_1^2} = \frac{2x_2}{x_1^2}, \qquad 0, x_2 < x_1,$$

zero elsewhere, where $0 < x_1 < 1$. The conditional mean of X_2, given $X_1 = 1/2$, is

$$E\left(X_2 \mid x_1 = \frac{1}{2}\right) = \int_0^{x_1} x_2\left(\frac{2x_2}{x_1^2}\right)dx_2 = \frac{2}{3}x_1, \qquad 0 < x_1 < 1.$$

i.e. $\frac{2}{3}\times\frac{1}{2} = \frac{1}{3} = 0.33$

47. **2**

$x_1 = \sqrt{2}$, $\qquad x_2 = \sqrt{2x_1} = \sqrt{2\sqrt{2}}$

$\therefore \qquad \sqrt{2} > 1 \;\; \therefore \;\; 2\sqrt{2} > 2 \Rightarrow \sqrt{2\sqrt{2}} > \sqrt{2}$

$\Rightarrow \qquad x_2 > x_1$ or $x_1 < x_2$

Now $\qquad x_{r-1} < x_r \Rightarrow \sqrt{2x_{r-1}} < \sqrt{2x_r}$

$\Rightarrow \qquad x_r < x_{r+1}$

by the mathematical induction

$x_n < x_{n+1} \;\; \forall n \in N$

$\Rightarrow \qquad \{x_n\}$ monotonically increasing ...(1)

again, $x_1 < 2$ Let $x_r < 2$

$\Rightarrow \qquad \sqrt{2x_r} < \sqrt{2\sqrt{2}} = 2$

$\Rightarrow \qquad x_{r+1} < 2$

again by the mathematical induction

$x_n < 2 \;\forall\; n \in N$ ...(2)

Thus sequence $\{x_n\}$ is bounded

(1) and (2) $\Rightarrow$ sequence $\{x_n\}$ is convergent

Let $\text{Lim}\, x_n = \xi$

$\Rightarrow \quad \text{Lim}\, x_n = \text{Lim}\, x_{n+1} = \xi$

$\therefore \quad x_{n+1} \sqrt{2x_n} = \Rightarrow \text{Lim}\, x_{n+1} = \sqrt{2 \text{Lim}\, x_n}$

$\Rightarrow \quad \xi = \sqrt{2\xi} \Rightarrow \xi^2 - 2\xi = 0$

$\Rightarrow \quad \xi = 0$ or $\xi = 2$ but $x \neq 0$ so $\xi = 2$.

48. **0.1**

The probability that the first ball drawn is black is $P_r\{E_1\} = 2/(3 + 2) = \frac{2}{5}$. The probability that the second ball drawn is black, given that the first ball drawn was black, is $P_r\{E_2 | E_1\} = 1/(3 + 1) = \frac{1}{4}$. Thus the probability that both balls drawn are black is

$$\Pr\{E_1 E_2\} = P_r\{E_2 | E_1\} = \frac{2}{5}.\frac{1}{4} = \frac{1}{10} = 0.1$$

49. 0.5049

We have P (head on exactly 50 coins) = P(head on exactly 51 coins)

$$\Rightarrow \quad {}^{100}C_{50} \times p^{50} (1-p)^{50} = {}^{100}C_{51} \times p^{51} (1-p)^{49}$$

$$\Rightarrow \quad p = \frac{51}{101} = 0.5049$$

50. 4

Binomial distribution is defined as

$$B(n; p; q) = {}^{10}c_x \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{10-x}; \qquad x = 0, 1, \ldots, 10$$

if we compare it by the binomial distribution then we can say that

$$n = 10,\ p = \frac{2}{5} \quad \text{and} \quad q = \frac{3}{5}$$

and mean of Binomial distribution is

$$E(x) = np = 10 \times \frac{2}{5} = 4$$

51. 12

Since X ~ P(1) and Y ~ P(3) as 1, 3 are means of random variable X and Y resp. and we know that mean of random variable which follows Poisson distribution is equal to parameter λ.

Now $E(X) = \lambda_1 = 1$

then $Var(X) = \lambda_1 = 1$

again $E(Y) = \lambda_2 = 3$

then $Var(Y) = \lambda_2 = 3$

So, $Var(3X + Y) = 9Var(X) + Var(Y)$ [$\because$ X, Y are independent]

$= 9(1) + 3 = 9 + 3 = 12$

52. 0.375

Let E denote the event that a six occurs and A the event that it is a six. We have P(E) = 1/6, P(E') = 5/6, P(A/E) = $\frac{3}{4}$ and P(A/E') = 1/4. By Baye's theorem

$$P(E/A) = \frac{P(E).P(A/E)}{P(E).P(A/E) + P(E').P(A/E')} = \frac{\left(\frac{1}{6}\right)\left(\frac{3}{4}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{4}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{4}\right)} = \frac{3}{8} = 0.375$$

53. 0.25

Let the lengths of three parts of the rod be x, y and $\ell - (x + y)$.

Then $x > 0$, $y > 0$ and $\ell - (x + y) > 0$

i.e., $x + y < \ell$ or $y < \ell - x$

Since, in a triangle, the sum of any two sides is greater than third side, so

$$x + y > \ell - (x + y)$$

$\Rightarrow \qquad y > \frac{\ell}{2} - x$

$x + \ell - (x + y) > y$

$\Rightarrow \qquad y < \frac{\ell}{2}$

$y + \ell - (x + y) > x$

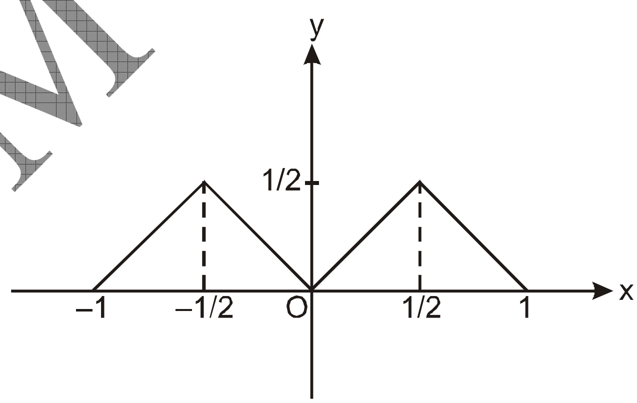
$\Rightarrow \qquad x < \frac{\ell}{2}$

$\Rightarrow \qquad \frac{\ell}{2} - x < y < \frac{\ell}{2}$ and $0 < x < \frac{\ell}{2}$

So, required probability $= \dfrac{\int_0^{\ell/2}\int_{\ell/2-x}^{\ell/2} dydx}{\int_0^{\ell}\int_0^{\ell-x} dydx} = \dfrac{\int_0^{\ell/2}\left\{\frac{\ell}{2} - \left(\frac{\ell}{2} - x\right)\right\}dx}{\int_0^{\ell}(\ell - x)\,dx} = \dfrac{\int_0^{\ell/2} xdx}{\int_0^{\ell}(\ell - x)\,dx} = \dfrac{\frac{\ell^2}{8}}{\frac{\ell^2}{2}} = \frac{1}{4} = 0.25$

54. 0.5

Graph of f(x) = min(| x |, | x – 1 |, | x + 1 |)

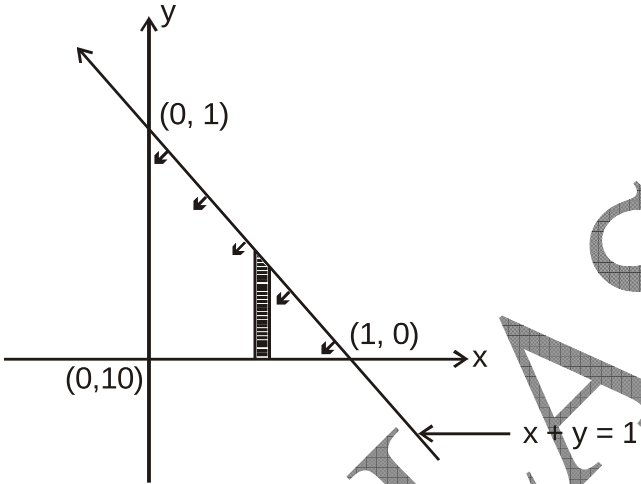


Required area $= 2 \times \frac{1}{2} \times 1 \times \frac{1}{2} = \frac{1}{2}$ sq unit = 0.5 sq unit

55. **0.0416**

From the figure it is clear that the limits of x and y are as x varies from 0 to 1 and y from 0 to $1 - x$

then



$$\iint xy\,dx\,dy = \int_0^1\int_0^{1-x} x\,y\,dx\,dy = \int_0^1\left[\frac{xy^2}{2}\right]_0^{1-x} dx = \int_0^1 \frac{x}{2}\left[(1-x)^2 - 0\right]dx = \frac{1}{2}\int_0^1 x(1-x)^2\,dx$$

$$= \frac{1}{2}\int_0^1\left[x - 2x^2 + x^3\right]dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] = \frac{1}{24} = 0.0416$$

56. **3**

Given differential equation is

$$\frac{d^{2y}}{dx^2} + 9y = 0 \quad \Rightarrow \quad (D^2 + 9)y = 0$$

$$\Rightarrow \quad (D + 3!)\,(D - 3i)y = 0$$

So, $y(x) = c_1\cos 3x + c_2\sin 3x$

which shows that two L.I. solutions of differential equation is cos 3x and sin 3x.

Now $W(x)=\begin{vmatrix}\cos 3x & \sin 3x\\ -3\sin 3x & 3\cos 3x\end{vmatrix}$ = $3\cos^2 3x + 3\sin^2 3x = 3$

i.e. W(x) = 3

which shows that wronskian does not depends on x.

i.e. w(1) = 3 = w(s).

57. 1.414

Given limit is

$$\lim_{x\to\infty}\frac{2x^{1/2}+3x^{1/3}+4x^{1/4}+......+nx^{1/n}}{(2x-3)^{1/2}+(2x-3)^{1/3}+........+(2x-3)^{1/n}}$$

if we divides numerator and denomenator by then we get

$$\lim_{x\to\infty}\frac{2+3x^{-1/6}+..........+nx^{1/n-1/2}}{\sqrt{2}\left[\left(1-\frac{3}{2x}\right)^{1/2}+\left(1-\frac{3}{2x}\right)^{1/3}+.........+\left(1-\frac{3}{2x}\right)^{1/n}\right]}$$ where n > 0

after applying limit we get the required result as

$$\frac{2}{\sqrt{2}}=\sqrt{2}=1.414$$

58. 0

We know that taylor's expansion is given as about origin is

$f(0 + x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+....$

Given f(0) = –1

$f'(0) = 1+\frac{1}{f(0)} = 1 - 1$ [$\because$ f(0) = –1]

= 0

$$f''(0) = \frac{-f(0)}{[f(0)]^2} = \frac{0}{[-1]^2} = \frac{0}{1} = 0$$

Thus, the coefficient of x^2 in the Taylor's expansion of f about zero is $\frac{f''(0)}{2!} = \frac{0}{2!} = 0$

59. 0.5

Given that q is parameter of binomial distribution i.e. $E(X) = n\theta$

Now

$$E = \left[\frac{X + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}}\right] = \frac{1}{n+\sqrt{n}} E\left[X + \frac{1}{2}\sqrt{n}\right] = \frac{1}{n+\sqrt{n}}\left[E(X) + \frac{1}{2}\sqrt{n}\right] = \frac{1}{n+\sqrt{n}}\left[n\theta + \frac{1}{2}\sqrt{n}\right]$$

By the observation we can say that

ie we put $\theta = \frac{1}{2}$ then

$$E\left[\frac{X + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}}\right] = \frac{1}{2} = \theta$$

which shows that the minimax estimator is unbiased estimator $\theta = \frac{1}{2}$ = 0.5.

60. 0.91

$$M_X(t) = e^{4(e^t - 1)}$$

which is the mgf of Poisson distribution with parameter $\lambda = 4$

Now $E(x) = \mu = \lambda = 4$

and $Var(x) = \sigma^2 = \lambda = 4$ i.e. $\boxed{\sigma = 2}$

then $P(\mu - 2\sigma < x < \mu + 2\sigma) = P(\mu - 4 < x < \mu + 4) = P(0 < x < 8) = \sum_{x=1}^{7} P(X = x)$

$$= \sum_{x=1}^{7} \frac{e^{-4}(4)^x}{x!} = e^{-4} \sum_{x=1}^{7} \frac{(4)^x}{x!} = 0.018\left[\frac{4^1}{1!} + \frac{4^2}{2!} + ... + \frac{4^7}{7!}\right] = 0.018[50.806] = 0.91$$