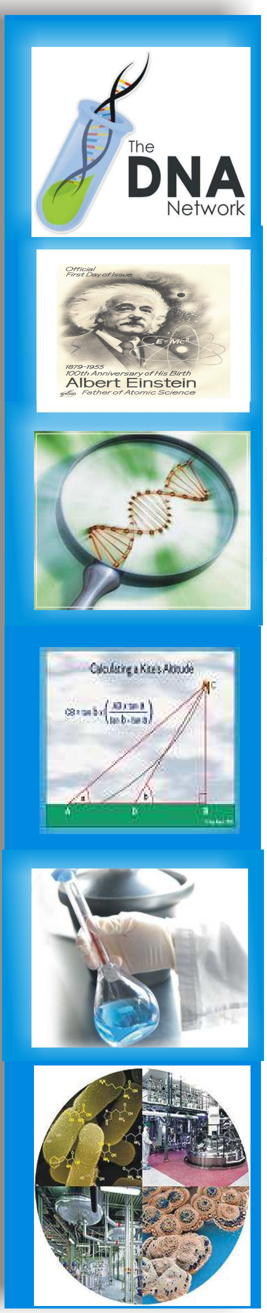


# IIT JAM - MATHEMATICS

## SAMPLE THEORY

- AREA OF THE CURVE
- GROUP AND GROUP OF PERMUTATION
- UNIFORM CONVERGENCE AND DIFFERENTIATION


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**Area by Double Integration**

Let the area ABCD be divided into sub-areas by drawing lines parallel to x and y-axis respectively such that the distance between two adjoining lines drawn parallel to y-axis be $\delta x$ and those drawn parallel to x-axis be $\delta y$.

Let P(x, y) and Q(x + $\delta x$, y + $\delta y$) be two neighbouring points on the curve AD whose equation is y = f(x) as in case (a). PN and QM are the ordinates at P and Q respectively. Then the area of the element, shown by shaded lines in adjoining figure is $\delta x \delta y$.

Consequently the area of the strip PNMQ = $\int_{y=0}^{f(x)} dxdy$, where y = f(x) is the equation of AD.

$\therefore$ The required area ABCD = $\int_{x=a}^{b}\int_{y=0}^{f(x)} dxdy$.

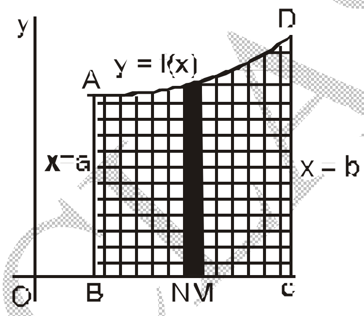


**Fig.1**

In a similar way, we can prove that the area bounded by the curve x = f(y), the y-axis and the abscissa at y = a and y = b is given by

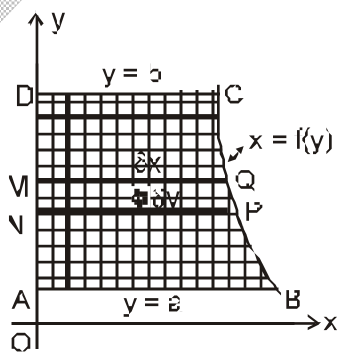


**Fig.2**

If we are to find the area bounded by the two curves $y = f_1(x)$ and $y = f_2(x)$ and the ordinates $x = a$ and $x = b$ i.e. the area ABCD in the figure below then the required area = $\int_{x=a}^{b}\int_{y=f_2(x)}^{f_1(x)} dxdy$.

**Ex.** Find the area of the region bounded by the parabolas $y = x^2$ and $y = 4 - x^2$.

**Sol.** $x^2 = y$ represents a parabola whose vertex is at (0, 0) and $y = 4 - x^2$ represents a parabola whose vertex is at (0, 4).

Solving the two equations we get $y = 4 - y$ or $2y = 4$ or $y = 2$

$\therefore x^2 = 2$ or $x = \pm\sqrt{2}$

$\therefore$ The two parabolas intersect at $A(\sqrt{2}, 2)$ and $B(-\sqrt{2}, 2)$.

$\therefore$ Required area = 2(area OACO) = 2[area OADO + area DACD] = $2\left[\int_{y=0}^{2} xdy + \int_{y=2}^{4} xdy\right]$,

(where the first integral is for $x^2 = y$ and second for $y = 4 - x^2$ )

$$= 2\left[\int_0^2 \sqrt{y}dy + \int_2^4 \sqrt{(4-y)}dy\right] = 2\left[\left(\frac{2}{3}y^{\frac{3}{2}}\right)_0^2 - \left\{\frac{2}{3}(4-y)^{\frac{3}{2}}\right\}_2^4\right]$$

$$= 2\left[\left(\frac{4}{3}\right)\sqrt{2} + \left(\frac{2}{3}\right)(2)^{\frac{3}{2}}\right] = \left(\frac{16}{3}\right)\sqrt{2}.$$ **Ans.**

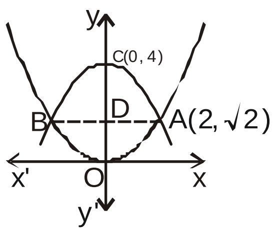


**Fig.3**

**Area of curve given by polar equation**

**(a) Single Integration**

The area bounded by the curve $r = f(\theta)$, where $f(\theta)$ is a single value continuous function of $\theta$ in the domain $(\alpha, \beta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$ is equal to $\frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta,\ (\alpha < \beta)$

**Proof.** Let O be the pole, OX the initial line and AB be the portion of the arc of the curve $r = f(\theta)$ between the radii vectors $\theta = \alpha$ and $\theta = \beta$.

Let $P(r, \theta)$ be any point on the curve between A and B. Let $Q(r + \delta r, \theta + \delta\theta)$ be a point neighbouring to P. Join OP and OQ. With $OP = r$ as radius and O as centre describe an arc PN of the circle meeting OQ in N. Similarly with O as centre and $OQ = r + \delta r$ as radius draw another arc QM of circle meeting OP produced in M.

Let the sectorial areas OAP and OAQ be denoted by S and $S + \delta S$ respectively.

Then the area $OPQ = (S + \delta S) - S = \delta S$.

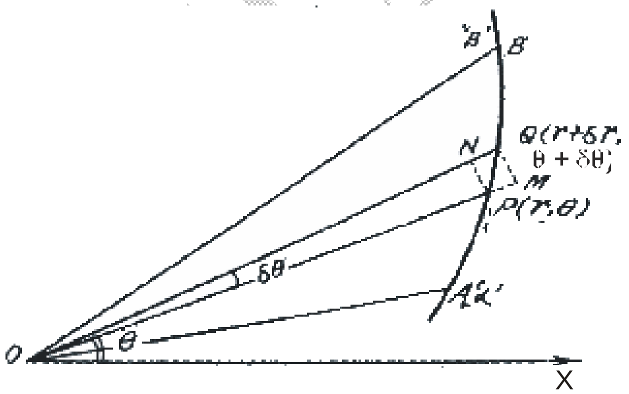


**Fig. 4**

Also as OP = r, OQ = r + $\delta r$ and $\angle POQ = \delta\theta$ so sectorial area OPN = $\frac{1}{2}r^2\delta\theta$ and sectorial area OQM = $\frac{1}{2}(r+\delta r)^2\delta\theta$. Now area OPQ lies between area OPN and area OQM i.e. Area $\delta S$ lies between $\frac{1}{2}r^2\delta\theta$ and $\frac{1}{2}(r+\delta r)^2\delta\theta$. i.e. $\left(\frac{\delta S}{\delta\theta}\right)$ lies between $\frac{1}{2}r^2$ and $\frac{1}{2}(r+\delta r)^2$

$\therefore$ In the limit as $\delta\theta \to 0$, we have $\left(\frac{dS}{d\theta}\right) = \frac{1}{2}r^2$ or $\frac{1}{2}r^2 d\theta = dS$

Integrating $\frac{1}{2}\int_\alpha^\beta r^2 d\theta = [\text{area } S]_\alpha^\beta$, = (Area S when $\theta = \beta$) – (Area S when $\theta = \alpha$) = (Area AOB) – (0) = Area AOB.

Hence, required area AOB = $\frac{1}{2}\int_\alpha^\beta r^2 d\theta$ .

**(b) Double Integration**

The area bounded by the curve r = f($\theta$), where f($\theta$) is single valued function of $\theta$ in the domain ($\alpha$, $\beta$) and radii vectors $\theta = \alpha$ and $\theta = \beta$ is $\int_{\theta=\alpha}^{\beta}\int_{r=0}^{f(\theta)} r\,d\theta\,dr$

**Ex.** Find by double integration the area lying inside the cardioid r = a(1 + cos$\theta$) and out side the circle r = a.

**Sol.** Required area = area ABCDA = 2(area ABDA) =

$$= a^2\int_0^{\frac{\pi}{2}}[(1+\cos\theta)^2 - 1]d\theta = a^2\int_0^{\frac{\pi}{2}}(\cos^2\theta + 2\cos\theta)d\theta = a^2\left[\frac{1}{2}\cdot\frac{\pi}{2} + 2(\sin\theta)_0^{\frac{\pi}{2}}\right] = a^2\left[\frac{\pi}{4} + 2\right]$$

$= \frac{1}{4}a^2(\pi + 8)$ **Ans.**

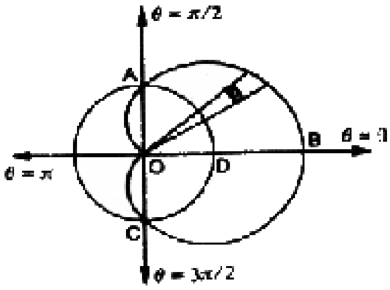


Fig 5

## GROUP

### Binary Operation

Let S be a non-empty set. Any function from S x S to S is called binary operation. i.e.

if o : s x s $\to$ s defined as$\circ$ (a,b) = a $\circ$ b $\in$ S, $\forall$a,$\in$ S, then is binary operation.

### Mathematical Structure

Let S be a non empty set. Let be an operation on S then (S,) is a mathematical structure.

### Grouped (Quasi - group)

Mathematical structure (S,) is said to be grouped, if is binary operation i.e., .

$\forall a,b \in S \Rightarrow a \circ b \in S$

### Semi group

A group (S, $\circ$), is semi group if it is associative i.e.,

**Monoid** $(a \circ b) \circ c = a \circ (b \circ c), \forall a,b,c \in S$

If identity element $e \in S$ exist in a semi group (S,), then it is monoid, i.e., $\forall a \in S, \exists e \in S : a \circ e = a = e \circ a$

### Group

If inverse element exists for every element in a monoid (S,), then it is a group, i.e., $\forall a \in S$, $\exists a^{-1} \in S : : a^{-1} = e \; = a^{-1} \circ a$

**Commutative group (Abelian Group)**

A group (S,), is a commutative group, if . $\forall a,b \in S, a \circ b = b \circ a$

**The Different Groups**

| | Quasi group | Semi group | Monoid | Group | Abelian Group |
|---|---|---|---|---|---|
| Clouser | ✓ | ✓ | ✓ | ✓ | ✓ |
| Associative | - | ✓ | ✓ | ✓ | ✓ |
| Existence of Identify | - | - | ✓ | ✓ | ✓ |
| Existence of inverse | - | - | - | ✓ | ✓ |
| Commutative | - | - | - | - | ✓ |

**Table - 1**

## GROUP

### Definition

Let $G \neq \phi$ be a set. Let be an operation defined in G, then mathematical structure $(G,\circ)$ will be group if it satisfies.

(i) Closure law : $\forall a,b \in G \Rightarrow a \circ b \in G$

(ii) Associative law: $(a \circ b) \circ c = a \circ (b \circ c), \forall a,b,c \in G$

(iii) Existence of identity: $\forall a \in G, \exists e \in G : a \circ e = a = e \circ a$

(iv) Existence of inverse: $\forall a \in G, \exists a^{-1} \in G : a \circ a^{-1} = e = a^{-1} \circ a$

### Results

- Identity element in a group is unique.
- Inverse of each element of a group is unique.
- If a,b $\in$ G, then $(ab)^{-1} = b^{-1} a^{-1}$. This law is known as reversal rule. One can generalize it as $(abc.....z)^{-1} = z^{-1}..... c^{-1}b^{-1}a^{-1}$.
- Cancellation law holds in a group. i.e. $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow c = a$
- If a,b $\in$ G, then linear equations $a \circ x = b, y \circ a = b$ have unique solutions for x, y$\in$ G

**Order of Group**

The number of elements in a group G, is order of group denoted by of O(G).

If (G, *) if an infinite group then it is said to be of infinite order.

**Order of Element**

Let G be a group. Let $a \in G$, then n is called order of element a, denoted by O(a) = n, if $a^n = e$, where n is least positive integer.

**Results on Order of an Element of a Group**

- The order of every element of a finite group is finite.
- If there is no positive integer n such that $a^n = e$, than order of a, o(a) is infinite or zero.
- The order of every element of a finite group is less than or equal to the order of the group.

If G is a finite then $o(a) \le 0(G), a \in G$ .

- The order of an element of a group is same as that of its inverse.
- Order of any integral power of an element $a \in G$ cannot exceed the order of a.
- If $a \in G$ a group ,o(a) = n and $a^m = e$, then n/m.
- If $a \in G$ is an element of order n and p is prime to n, then $a^p$ is also of order n.
- If every element of a group except the identity element is of order two, then G is abelian.
- If every element of a group G is its own inverse, then G is abelian.

**Theorem .** If order of an element a of a group (G,*) is n then $a^m = e$, iff m is a multiple of n.

**Proof.** Let $a^m = e$

By division algorithm $m = nq + r,\ 0 \le r \le n$ where $q, r \in Z$

$$\therefore \quad a^m \Rightarrow a^{mq+r} = e$$

$$\Rightarrow a^{nq}.a^r = e$$

$$\Rightarrow (a^n)^q.a^r = e \quad \left[\therefore (a^m)^n = a^{mn}\right]$$

$$\Rightarrow e^q.a^r = e \quad \left[\therefore O(a) = n \Rightarrow a^n = e\right]$$

$\Rightarrow a^r = e$

$\Rightarrow r = 0 \qquad [\therefore 0 \le r \le n]$

$\Rightarrow m = mq$

So, n/m

**Conversely**

Let m is multiple of n i.e. $m = nq (q \in Z)$

$m = nq \Rightarrow a^m = a^{nq} = (a^n)^q = e^q = e$

So, $a^m = e \Leftrightarrow$ m is multiple of O(a).

If a, $x \in G$ a group, then $O(a) = O(x^{-1}ax)$

**Theorem .** For any element a of group of G:

$O(a) = O(x^{-1}ax), \ \forall \ x \in G$

**Proof.** Let $a \in G, x \in G$

$(x^{-1}ax)^2 = (x^{-1}ax)(x^{-1}ax)$

$= x^{-1}(xx^{-1})ax$

$= x^{-1}$ aeax

$= x^{-1}(aea)\, x$

$= x^{-1}a^2x$

Again consider that $(x^{-1}ax)^{n-1} = x^{-1}a^{n-1}x$, where $(n - 1) \in N$

$\Rightarrow (x^{-1}ax)^{n-1}(x^{-1}ax) = (x^{-1}a^{n-1}x)(x^{-1}ax)$

$\Rightarrow (x^{-1}ax)^n = x^{-1}a^{n-1}(xx^{-1})ax$

$= x^{-1}a^{n-1}(eax)$

$= x^{-1}(a^{n-1}a)x = x^{-1}a^nx$

By mathematical induction

$\left(x^{-1}ax\right)^n = x^{-1}a^n x, \forall\, n \in N$

now let $O(a) = n$ and $O(x^{-1}ax) = m$

$(x^{-1}ax)^n = x^{-1}a^n x = x^{-1}ex = e$

$\Rightarrow O\left(x^{-1}ax\right) \le n$

$\Rightarrow m \le n$ (1)

Again $O(x^{-1}ax) = m \Rightarrow \left(x^{-1}ax\right)^m = e$

$\Rightarrow x^{-1}a^m x = e$

$\Rightarrow x\left(x^{-1}a^m x\right)x^{-1} = xex^{-1} = e$

$\Rightarrow \left(xx^{-1}\right)a^m\left(xx^{-1}\right) = e$

$\Rightarrow ea^m e = e$

$\Rightarrow O(a) \le m$

$\Rightarrow n \le m$ (2)

By (1) and (2) $n = m$

$\Rightarrow O(a) = O\left(x^{-1}ax\right)$

If $O(a)$ is infinite then $O(x^{-1}ax)$ is also infinite.

**Ex.** If a,b are elements of an abelian group G, then prove that :

$(ab)^n = a^n b^n, \ \forall\ n \in Z$

**Sol. Case (i)** When $n = 0$

$(ab)^0 = e = ee$ [By the Def$^n$]

$= a^0 b^0$

**Case (ii)** When $n > 0$;

$(ab)^1 = ab = a^1 b^1$

Result is true for $n = 1$

Let result is true for $n = K$

$(ab)^k = a^k b^k$

$\Rightarrow (ab)(ab)^k = (ab)(a^k b^k)$

$\Rightarrow (ab)^{k+1} = a(ba^k)b^k$ [associativity]

$= a(a^k b)b^k$

$= (aa^k)(bb^k)$

$= a^{k+1} b^{k+1}$

By mathematical induction result is true for all integers

**Case (iii)** When $n < 0$ Let $n = -m$ where $m \in Z^+$

$(ab)^n = (ab)^{-m} = [(ab)^m]^{-1}$

$= (a^m b^m)^{-1}$

$= (b^m a^m)^{-1}$

$= (a^m)^{-1} (b^m)^{-1}$

$= a^{-m} b^{-m}$

$= a^n b^n$

By above conditions

G is Commutative $\Rightarrow (ab)^n = a^n b^n,\ \forall n \in Z$

**Permutation**

Let P be a finite set having n distinct elements. Then a one-one mapping onto itself $f : P \to P$ is called a permutation of degree n, in the finite set P is called the degree of the permutation.

Let $P = \{a_1, a_2, a_3\}$ be a finite set having n distinct elements. If $f : P \to P$ is one - one onto, then f is a permutation of degree n. Let f is a permutation of degree n.

Let $f(a_1) = b_1, f(a_2) = b_2, \ldots\ldots f(a_n) = b_n$ symbolically one can write it as

$f = \begin{pmatrix} a_1 & a_2 & \ldots\ldots & a_n \\ b_1 & b_2 & \ldots\ldots & b_n \end{pmatrix}$, where each element in the second row is f image of the elements of the first row.

**Equality of two permutations**

Two permutations $f_1$ and $f_2$ on P are said to be equal. If we have $f_1(a) = f_2(a)$.

**Total number of distinct Permutations P**

Let P be a finite set having n distinct elements. There shall be n! permutations of degree n, of the element in a set.

**Identity Permutations**

If I is a permutation of degree n such that I replace each element by itself, I is called the identity permutation of degree n.

**Inverse of a Permutation**

If f is a permutation of degree n defined on a finite non-empty set P. Since f is one-one onto, it is inverse able.

$$f = \begin{pmatrix} a_1 & a_2 & \ldots\ldots & a_n \\ b_1 & b_2 & \ldots\ldots & b_n \end{pmatrix} \text{then } f^{-1} = \begin{pmatrix} b_1 & b_2 & \ldots\ldots & b_n \\ a_1 & a_2 & \ldots\ldots & a_n \end{pmatrix}$$

- $f^{-1}$ is obtained by interchanging the rows of f because $f(a_1) = b_1$. $\Rightarrow f^{-1}(b_1) = a_1$

**Products or composite of permutations**

If two permutations of degree n be

$$f_1 = \begin{pmatrix} a_1 & a_2 & \ldots\ldots & a_n \\ b_1 & b_2 & \ldots\ldots & b_n \end{pmatrix} \text{and } f_2 = \begin{pmatrix} b_1 & b_2 & \ldots\ldots & b_n \\ c_1 & c_2 & \ldots\ldots & c_n \end{pmatrix}$$

Then the products of these two functions is defined as

$$f_1 f_2 = \begin{pmatrix} a_1 & a_2 & \ldots\ldots & a_n \\ c_1 & c_2 & \ldots\ldots & c_n \end{pmatrix}$$

- The product $f_1\, f_2$ is also a permutation of degree n.
- Product of permutations is not necessarily commutative.

**Associativity of permutation.** The associative law is true for the product of the permutations i.e. f, g and h are permutations, then (fg)h = f(gh)

**Group of Permutations**

The set of all the permutations of a given non-empty set A is denoted by $S_A$. Therefore if A = {a, b}, then

$$S_A = \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}$$

If A = {a, b, c}, then

$$S_A = \left\{ \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}, \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \right\}$$

It can be easily seen that

$$O(A) = n \Rightarrow O(S_A) = n!$$

**Even and odd permutation**

A permutation is said to be an even permutation if it can be expressed as a product of an even number of transposition.

- A permutation can not be both even or odd i.e, permutation f is expressed as a product of transposition, then the number of transposition is either always even or always odd.
- Identity permutation is always an even permutation.
- The product of two even permutation is an even permutation.
- The product of two odd permutations is an even permutation.
- A cycle of length n can be expressed as the product of n-1 permutation.
- The inverse of an even permutation is an even permutation and the inverse of an odd permutation is an odd permutation.
- Out of n! permutations on n symbols $\frac{1}{2}n!$ are even $\frac{1}{2}n!$ are odd.

**Alternating group. (Group of even permutation).**

On the basis of the above conclusions of the product of even and odd permutations of any set, we will show that the set of permutations is also a group.

**Theorem .** The set $A_n$ of all even permutations of degree n is a group of order $\frac{1}{2}n!$ for the product of permutations.

**Important Results**

(i) When n = 3, $A_3 = \{(1), (1\ 2\ 3), (1\ 3\ 2)\}$

(ii) $A_n$ is a simple group for $n \geq 5$

Every group of prime order is a simple group because such group has no proper subgroup.

(iii) The set of odd permutations of degree n is not a group because it is not closed for multiplication.

(iv) If H is a sub group of G and $N \triangleleft G$, then $H \cap N$ need not be normal in G.

For example, let

$N = A_4 = \{(1), (123), (124), (132), (134), (142), (143), (234), (243), (12)\ (34), (13)\ (24), (14)\ (23)\}$

$H = \{(1), (1234), (13)\ (24), (1432), (12)\ (34), (14)(23), (13)\ (24)\}$

This can be easily verified that

$N \triangleleft S_4$ and H is a subgroup of $S_4$.

But $H \cap N$ is not a normal subgroup of $S_4$

(v) $\frac{S_3}{A_3}$ is a commutative and cyclic group, being group of order 2 but $S_3$ is non abelian and not a cyclic group.

(vi). The alternating group $A_n$ of all even permutations of degree n is a normal subgroup of the symmetric group $S_n$.

i.e. $A_n \triangleleft S_n$

**Ex.** If

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix}, \ \sigma = (1\ 3\ 4)\ (5\ 6)\ (2\ 7\ 8\ 9)$$

then find $\sigma^{-1}\rho\sigma$ and by expressing the permutation $\rho$ as the product of disjoint cycles, find whether $\rho$ is an even permutation or odd permutation. Also find its order.

**Sol.** $\sigma = (1\ 3\ 4)\ (5\ 6)\ (2\ 7\ 8\ 9)$

$$= \begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 2 & 7 & 8 & 9 \\ 3 & 4 & 1 & 6 & 5 & 7 & 8 & 9 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{pmatrix}$$

$$\therefore \quad \sigma^{-1} = \begin{pmatrix} 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 3 & 6 & 5 & 2 & 7 & 8 \end{pmatrix} \quad \text{....(1)}$$

$$\text{Again } \rho\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 4 & 1 & 6 & 5 & 8 & 9 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix} \quad \text{...(2)}$$

$\sigma^{-1}\rho\sigma = \sigma^{-1}(\rho\sigma)$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 3 & 6 & 5 & 2 & 7 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \\ 8 & 9 & 5 & 2 & 6 & 3 & 1 & 4 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 2 & 6 & 7 & 5 & 4 & 3 & 1 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 5 & 2 & 6 & 3 & 1 & 4 & 7 \end{pmatrix} = (1\ 8\ 4\ 2\ 9\ 7)\ (3\ 5\ 6)$$

$$\text{Again } \rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \end{pmatrix} = (1\ 7\ 2\ 8\ 3\ 9)\ (4\ 6\ 5)$$

$= (1\ 9)\ (1\ 3)\ (1\ 8)\ (1\ 2)\ (1\ 7)\ (4\ 5)\ (4\ 6)$

= product of 7 (odd) transpositions.

Since $\rho$ is equal to the product of odd transpositions,

Therefore this is a odd permutation.

Finally, O(p) = L. C. M. of {O(1 7 2 8 3 9), O(4 6 5)}

= L. C. M. of {6, 3} = 6

## Uniform convergence of sequences

Suppose that the sequence $\{f_n(x)\}$ converges for every point x in R. It means that the function $f_n$ tends to a definite limit as $n \to \infty$ for every x in R. This limit will be a function of x, say f. Then from the definition of a limit it follows that for every $\in > 0$, there exists a positive integer m such that

$$n \geq m \Rightarrow |f_n(x) - f(x)| < \in .$$

The integer m will depend upon x as well as $\in$ and so we may write it symbolically as m(x, $\in$). Now suppose that we keep $\in$ fixed and vary x. Then for a given point x in R, there will correspond a value of m (x, $\in$). In this way, we shall get a set of values of m(x, $\in$). This set of values of m(x, $\in$) may or may not have an upper bound. If this set has an upper bound, say M, then for every point x in R, we have

$$n \geq M \Rightarrow |f_n(x) - f(x)| < \in .$$

In such a case, we say that the sequence $\{f_n\}$ converge uniformly to f on X.

**Definition.** A sequence $\{f_n\}$ of functions is said to converge uniformly on R to a function f if for every $\in > 0$, there can be found a positive integer m such that

$$n \geq m \Rightarrow |f_n(x) - f(x)| < \in$$

for all $x \in R$.

**Remark.** Observe that the convergence of a sequence $\{f_n(x)\}$ at every point (i.e., point-wise convergence) does not necessarily ensure its uniform convergence on R. A sequence of functions may be convergent at every point of R and yet may not be uniformly convergent on R. For example, consider the sequence $\{f_n\}$ defined on [0, 1] as follows by $f_n(x) = x^n$.

Here, we have $\lim_{n\to\infty} x^n = 0$ if $0 \leq x < 1$

and $\lim_{n\to\infty} x^n = 1$ if $x = 1$.

Thus the limit function f is defined by

$$f(x)\begin{cases}0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad x = 1.\end{cases}$$

The function $f_n$ therefore has a definite limit for every value of x in [0, 1] as $n \to \infty$ and consequently the sequence $\{f_n(x)\}$ converges for every $x \in [0, 1]$.

to see whether the convergence is uniform, we consider the interval [0, 1]. Let $\in > 0$ be given. Then

$$|f_n(x) - f(x)| < \in \Rightarrow |x^n - 0| < \in \Rightarrow x_n < \in \Rightarrow \frac{1}{x^n} > \frac{1}{\in}$$

$$\Rightarrow n\log\frac{1}{x} > \log\frac{1}{\in} \Rightarrow n > \frac{\log(1/\in)}{\log(1/x)} \qquad ...(1)$$

Thus when $x \ne 1$, $m(x, \in)$ is any integer greater than

$\log(1/\in)/\log(1/x)$.

In particular $m(x, \in) = 1$ when x = 0.

Now as x, starting from 0, increases and approaches 1, it is evident from (1) that $n \to \infty$ and so it is not possible to determine a positive integer m such that

$$n \ge m \Rightarrow |f_n(x - f(x))| < \in$$

for all $x \in [0, 1[$.

Thus $\{f_n\}$ is not uniformly convergent in [0, 1[.

If, however, we consider the interval $0 \le x \le k$, where $0 < k < 1$, we see that the greatest value of $\log(1/\in)/\log(1/x)$ is $\log(1/\in).\log(1/k)$ so that if we take m equal to any positive integer greater than this greatest value, we have

$$n \ge m \Rightarrow |f_n(x) - f(x)| < \in$$

for all $x \in [0, k]$

Thus $\{fn(x)\}$ converges unformly on [0, k].

## Uniform Convergence and Differentiation.

**Theorem.** Let $\{f_n\}$ be a sequence of the real valued functions defined on [a, b] such that

(i) $f_n$ is differentiable on [a, b] for n = 1, 2, 3, ....,

(ii) The sequence $\{f_n(c)\}$ converges for some point c of [a, b],

(iii) The sequence $\{f'_n\}$ converges uniformly on [a, b].

Then the sequence $\{f_n\}$ converges uniformly to a differentiable limit f and

$$\lim_{x\to\infty} f_n'(x) = f'(x) \qquad (a \le c \le b).$$

**Proof.** Let $\in > 0$ be given. Then by the convergence of $\{f_n(c)\}$ and by the uniform convergence of $\{f_n'\}$ on [a, b], there exists a positive integer m such that for all $n \ge m$, $p \ge m$,

we have $\qquad |f_n(c) - f_p(c)| < \frac{\in}{2}$ ...(1)

and $\qquad |f_n'(x) - f_p'(x)| < \frac{\in}{2(b-a)} \quad (a \le x \le b)$. ...(2)

Applying the mean value theorem of differential calculus to the function $f_n - f_p$, we have

$$[f_n(x) - f_p(x)] - [f_n(y) - f_p(y)] = (x - y)\,[f_n'(\xi) - f_p'(\xi)]$$

For any x and y in [a, b] and for some $\xi$ between x and y provided $n \ge m$, $p \ge m$. Hence

$$|f_n(x) - f_p(x) - f_n(y) + f_p(y)| = |x - y|\,|f_n'(\xi) - f_p'(\xi)|$$

$$< \frac{|x-y|\in}{2(b-a)} \text{ by (2)} \qquad ...(3)$$

$$< \frac{\in}{2}\left[\because |x-y| \le (b-a)\right] \qquad ... (4)$$

for all $n, p \ge m$ and all $x, y \in [a, b]$. Now

$$|f_n(x) - f_p(x)| = |f_n(x) - f_p(x) - f_n(c) + f_p(c) + f_n(c) - f_p(c)|$$

$$\le |f_n(x) - f_p(x) - f_n(c) + f_p(c)| + |f_n(c) - f_p(c)$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in \qquad \text{by (1) and (4),}$$

for all $n, p \ge m$ and for all $x \in [a, b]$. Thus we have shown that given $\in > 0$, there exists a positive integer m such that

$$n \ge m,\ p \ge m,\ x \in [a, b] \quad \Rightarrow \quad |f_n(x) - f_p(x)| < \in.$$

It follows that the sequence $\{f_n\}$ converges uniformly to a function f and so

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad (a \le x \le b).$$

This proves the first result.

Now for an arbitrary but for the moment a fixed $x \in [a, b]$, define

$$F_n(y) = \frac{f_n(y) - f_n(x)}{y - x} \qquad F(y) = \frac{f(y) - f(x)}{y - x} \quad , \quad , \qquad \text{...(5)}$$

for $a \le y \le b$, $y \ne x$. Then

$$\lim_{y\to x} F_n(y) = \lim_{y\to x} \frac{f_n(y) - f_n(x)}{y - x} = f_n'(x) \qquad \text{...(6)}$$

for n = 1, 2, 3, ....

Now for $n \ge m$, $p \ge m$, we have

$$|F_n(y) - F_p(y)| = \left| \frac{f_n(y) - f_n(x) + f_p(y) - f_p(x)}{y - x} \right|$$

$$< \frac{\in}{2(b-a)} \qquad \text{by (3).}$$

It follows that $\{F_n\}$ converges uniformly for $y \ne x$. Since $\{f_n\}$ converges to f, we conclude from (5) that

$$\lim_{n\to\infty} F_n(y) = \lim_{n\to\infty} \frac{f_n(y) - f_n(x)}{y - x} = \frac{f(y) - f(x)}{y - x} = F(y) \qquad \text{...(7)}$$

Uniformly for $a \le y \le b$, $y \ne x$.

$$\lim_{y\to x} F(y) = \lim_{n\to\infty} f_n'(x)$$

or $$\lim_{y\to x} \frac{f(y) - f(x)}{y - x} = \lim_{n\to\infty} f_n'(x) \text{ by (5)}$$

or $$f'(x) = \lim_{n\to\infty} f_n'(x) \qquad \text{...(8)}$$

for every $x \in [a, b]$.

The theorem is thus completely established.

**Term by term differentiation.**

**Cor.** Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued differentiable functions on [a, b] such that $\sum_{n=1}^{\infty} u_n(c)$ converges for some point c of [a, b] and $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on [a, b].

Then the series $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on [a, b] to a differentiable sum function f and

$$, f'(x) = \lim_{n\to\infty} \sum_{m=1}^{n} u'_m(x) \quad (a \le x \le b).$$

In other words, if $a \le x \le b$, then

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty}\left[\frac{d}{dx}u_n(x)\right]$$

**Proof.** Let $f_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$.

Then $f_n'(x) = u_1'(x) + u_2'(x) + ... + u_n'(x)$

[$\because$ The differential coefficient of the sum of a finite number of differentiable functions is equal to the sum of their differential coefficients].

Hence the series $\sum_{n=1}^{\infty} u_n(x)$ and $\sum_{n=1}^{\infty} u'_n(x)$ are respectively equivalent to the sequences $\{f_n\}$ and $\{f_n'\}$. Now proceed as in the above theorem.

**Theorem .** Let $\{f_n\}$ be a sequence of real valued functions defined on [a, b] such that

(i) $f_n$ is differentiable on [a, b] for n = 1, 2, 3, ...

(ii) the sequence $\{f_n\}$ converges to f on [a, b],

(iii) the sequence $\{f_n'\}$ converges uniformly on [a, b] to g,

(iv) each $f_n'$ is continuous on [a, b].

Then $g(x) = f'(x)$ $(a \le x \le b)$. That is,

$$\lim_{n\to\infty} f_n'(x) = f'(x) \quad (a \le x \le b).$$

**Proof.** Since $\{f_n'\}$ is a uniformly convergent sequence of continuous functions, it follows that g is continuous on [a, b]. Moreover $\{f_n'\}$ converges uniformly to g on [a, x] where x is any point of [a, b]. Then we have

$$\lim_{n\to\infty}\int_a^x f_n'(t)\,dt=\int_a^x g(t)\,dt \qquad ...(1)$$

But by the fundamental theorem of integral calculus, we have

$$\int_a^x f_n'(t)\,dt=f_n(x)-f_n(a).$$

Also by hypothesis,

$$\lim_{n\to\infty} f_n(x)=f(x) \quad \text{and} \quad \lim_{n\to\infty} f_n(a)=f(a).$$

Hence (1) gives

$$f(x)-f(a)=\int_a^x g(t)\,dt \quad (a\le x\le b).$$

It follows

$$f'(x) = g(x) \quad (a\le x\le b)$$

or $\quad f'(x)=\lim_{n\to\infty} f_n'(x)$

**Ex.** Consider the series $\sum\frac{(-1)^{n-1}}{(n+x^2)}$ for uniform convergence for all values of x.

**Sol.** Let $u_n = (-1)^{n-1}$, $v_n(x)=\frac{1}{n+x^2}$.

Since $f_n(x)=\sum_{r=1}^{n} u_r = 0$ or 1 according as n is even or odd, $f_n(x)$ is bounded for all n.

Also $v_n(x)$ is a positive monotonic decreasing sequence converging to zero for all real values of x.

Hence the given series is uniformly convergent for all real values of x.

**Ex.** If $f(x)=\sum_1^{\infty}\frac{1}{n^3+n^4x^2}$, then find its differential coefficient

(A) $-2x\sum_1^{\infty}\frac{1}{n^2(1+nx^2)^2}$ (B) $2x\sum_1^{\infty}\frac{1}{n^2(1+nx^2)^2}$ (C) $\sum_1^{\infty}\frac{1}{n^2(1+nx^2)^2}$ (D) $\sum_1^{\infty}\frac{-1}{n^2(1+nx^2)^2}$

**Sol.** Here $\quad u_n(x)=\frac{1}{n^3+n^4x^2}$

and $u_n'(x) = \frac{2x}{n^2(1+nx^2)^2}$.

Now $u_n'$ (x) is maximum when $\frac{du_n'(x)}{dx} = 0$

i.e. $(1 + nx^2)^2 - 4nx^2(1 + nx^2) = 0$

or $1 - 3nx^2 = 0$ or $x = \frac{1}{\sqrt{(3n)}}$.

$\therefore$ Max. $|u_n'(x)| = \frac{2}{\sqrt{3n^{5/2}\left(1+\frac{1}{3}\right)^2}} = \frac{3\sqrt{3}}{8n^{5/2}}$.

Then $|u_n'(x)| < \frac{1}{n^{5/2}}$ for all values of x.

But $\sum \frac{1}{n^{5/2}}$ is convergent.

Hence by Weierstrass's M-test, the series $\Sigma u_n'$ is uniformly convergent for all real values of x. The term by term differentiation is therefore justified.

Hence . $f'(x) = \sum_{n=1}^{\infty} u_n'(x) = -2x\sum_{1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$