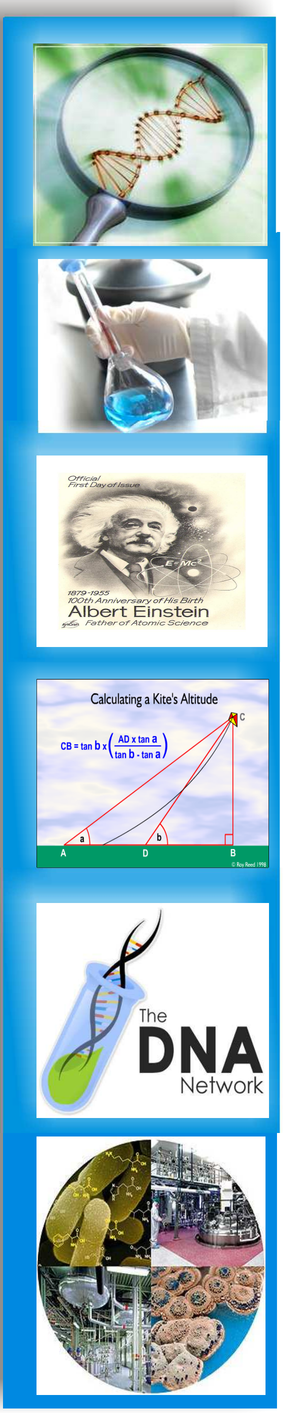

GATE SCIENCE - MATHEMATICS

SAMPLE THEORY

SEQUENCES , SERIES AND LIMIT POINTS OF SEQUENCES

- ***SEQUENCES***
- ***LIMITS : INFERIOR & SUPERIOR***
- ***ALGEBRA OF SEQUENCES***
- ***SEQUENCE TESTS***
- ***FOURIER SERIES***
- ***SOME PROBLEMS***

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1. SEQUENCE

A sequence in a set S is a function whose domain is the set N of natural numbers and whose range is a subset of S. A sequence whose range is a subset of R is called a real sequence.

$S_n = u_1 + u_2 + u_3 + ... + u_n$

$S_1 = u_1$

$S_2 = u_1 + u_2$

$S_3 = u_1 + u_2 + u_3$

…

…

$S_n = u_1 + u_2 + u_3 + ... + u_n \rightarrow$ series

$\downarrow$

Sequence

Bounded Sequence: A sequence is said to be bounded if and only if its range is bounded. Thus a sequence S_n is bounded if there exists

$k \le S_n \le K,\ \forall n \in N$

$\Leftrightarrow S_n \in [k, K]$

The l. u. b (Supremum) and the g.l.b (infimum) of the range of a bounded sequence may be referred as its g.l.b and l.u.b respectively.

2. LIMITS INFERIOR AND SUPERIOR

From the definition of limit, it follows that the limiting behavior of any sequence $\{a_n\}$ of real numbers, depends only on sets of the form $\{a_n : n \ge m\}$, i.e., $\{a_m, a_{m+1}, a_{m+2}, . . .\}$. In this regard we make the following definition.

Definition Let $\{a_n\}$ be a sequence of real numbers (not necessarily bounded). We define

$$\lim_{n\to\infty} \inf a_n = \sup_n \inf \{a_n, a_{n+1}, a_{n+2}, . . .\}$$

And $$\lim_{n\to\infty} \sup a_n = \inf_n \sup \{a_n, a_{n+1}, a_{n+2}, . . .\}$$

As the limit inferior and limit superior respectively of the sequence $\{a_n\}$.

Limit inferior and limit superior of $\{a_n\}$ is denoted by $\underline{\lim}_{n\to\infty} a_n$ and $\overline{\lim}_{n\to\infty} a_n$ or simply by $\underline{\lim}\ a_n$ and $\overline{\lim}\ a_n$ respectively.

We use the following notations for the sequence $\{a_n\}$, for each $n \in N$

$$\underline{A}_n = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\},$$

And $$\overline{A}_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$

Therefore, we have

$$\underline{\lim}\, a_n = \sup_n \underline{A}_n$$

And $$\overline{\lim}\, a_n = \inf_n \overline{A}_n$$

Now $\{a_{n+1}, a_{n+2}, \ldots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \ldots\}$, Therefore by taking infimum and supremum respectively, it follows that

$$\underline{A}_{n+1} \geq \underline{A}_n \text{ And } \overline{A}_{n+1} \leq \overline{A}_n$$

This is true for each $n \in \mathbf{N}$.

The above inequalities show that the associated sequences $\{\underline{A}_n\}$ and $\{\overline{A}_n\}$ monotonically increase and decrease respectively with n.

Remark: It should be noted that both limits inferior and superior exist uniquely (finite or infinite) for all real sequences.

Theorem: If $\{a_n\}$ is any sequence, then

$$\underline{\lim}\,(-a_n) = -\overline{\lim}\, a_n, \text{ and } \overline{\lim}\,(-a_n) = -\underline{\lim}\, a_n.$$

Let $b_n = -a_n$, $n \in N$ then we have

$$\underline{B}_n = \inf\{b_n, b_{n+1}, \ldots\}$$

$$= -\sup\{a_n, a_{n+1}, \ldots\} = -\overline{A}_n$$

And so

$$\underline{\lim}\,(-a_n) = \underline{\lim}\, b_n = \sup(\underline{B}_1, \underline{B}_2, \ldots\ldots)$$

$$= \sup\{-\overline{A}_1, -\overline{A}_2, \ldots\}$$

$$= -\inf\{\overline{A}_1, \overline{A}_2, \ldots\}$$

$$= -\inf \overline{A}_n = -\overline{\lim}\, a_n.$$

Also,

$$\underline{\lim}\, a_n = \underline{\lim}\,(-(a_n)) = -\overline{\lim}\,(-a_n).$$

Theorem: If $\{a_n\}$ is any sequence, then

$\underline{\lim}\, a_n = -\infty$ if and only if $\{a_n\}$ is not bounded below,

And $\overline{\lim}\, a_n = +\infty$ if and only if $\{a_n\}$ is not bounded above.

Let $\underline{A}_n = \inf \{a_n, a_{n+1}, \ldots\}$,

And $\bar{A}_n = \sup \{a_n, a_{n+1}, \ldots\}, n \in N$

By definition we have

$$\underline{\lim}\, a_n = -\infty \Leftrightarrow \sup \{\underline{A}_1, \underline{A}_2, \ldots\} = -\infty$$

$\Leftrightarrow$ $\underline{A}_n = -\infty, \quad \forall\ n \in \mathbf{N}$

$\Leftrightarrow$ $\inf \{a_n, a_{n+1}, \ldots\} = -\infty, \ \forall\ n \in \mathbf{N}$

$\Leftrightarrow$ $\{a_n\}$ is not bounded below:

The proof for limit superior is similar.

Corollary: If $\{a_n\}$ is any sequence, then

(i) $-\infty < \underline{\lim}\, a_n \le +\infty$ iff $\{a_n\}$ is bounded below.

and

(ii) $-\infty \le \overline{\lim}\, a_n < +\infty$ iff $\{a_n\}$ is bounded above.

For bounded sequences, we have the following useful criteria for limits inferior and superior respectively.

Limit points of a sequence.

A number ξ is said to be a limit point of a sequence S_n if given any nbd of ξ, S_n belongs to the same for an infinite number of values of n.

Now $\{S_{n+1}\ S_{n+2}, S_{n+3}, \ldots\} \subseteq \{S_n, S_{n+1}, S_{n+2}, \ldots\}$, therefore by taking infimum and supremum respectively, if follows that $\underline{A_{n+1}} \ge \underline{A_n}$ and $\overline{A_{n+1}} \le \overline{A_n}$ for each $n \in N$

Remark: Both limits inferior and superior exist uniquely (finite or infinite) for all real sequence.

Theorem: If $\{S_n\}$ is any sequence, then

$\inf S_n \le \underline{\lim}\, S_n \le \operatorname{Sup} S_n$

If $\{S_n\}$ is any sequence, then

$\underline{\lim}\{-S_n\} = -\overline{\lim}\, S_n$

And $-\overline{\lim}\, \{-S_n\} = \overline{\lim}\, S_n$

3. SOME IMPORTANT PROPERTIES OF ALGEBRA OF SEQUENCES

1. If $\{a_n\}$ is a bounded sequence such that $a_n > 0$ for all $n \in N$, then

(i) $\underline{\lim}\left(\dfrac{1}{a_n}\right) = \dfrac{1}{\overline{\overline{\lim}}\, a_n}$, if $\overline{\lim}\, a_n > 0$

(ii) $\underline{\lim}\left(\frac{1}{a_n}\right) = \frac{1}{\underline{\lim}\ a_n}$, if $\underline{\lim}\, a_n > 0$

2. If $\{a_n\}$ and $\{b_n\}$ are bounded sequence, $a_n \ge 0, b_n > 0$ for all $n \in N$, then

(i) $\underline{\lim}\left(\frac{a_n}{b_n}\right) \ge \frac{\underline{\lim}\, a_n}{\overline{\lim}\ b_n}$, if $\overline{\lim}\ b_n > 0$

(ii) $\overline{\lim}\left(\frac{a_n}{b_n}\right) \le \frac{\overline{\lim}\, a_n}{\underline{\lim}\ b_n}$, if $\underline{\lim}\ b_n > 0$

4. SOME IMPORTANT SEQUENCE TESTS

1. Cauchy's root test

Let Σu_n be +ve term series and

$$\lim_{n\to\infty}\{u_n\}^{u_n} = \ell$$

Then the series is

(i) Cgt if $\ell < 1$

(ii) Dgt if $\ell > 1$

(iii) No firm decision is possible if $\ell = 1$

2. Raabe's test

Let Σu_n be a +ve term series and

$$\lim n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \ell$$

then the series is

(i) Cgt if $\ell > 1$

(ii) Dgt if $\ell < 1$

(iii) No firm decision is possible if $\ell = 1$

3. Logarithmic Test:

If Σu_n is +ve terms series such that

$$\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) = \ell$$

Then the series

(i) cgt if $\ell > 1$

(ii) dgt if $\ell < 1$

4. Absolute convergent

A series Σu_n is said to be absolutely cgt if the positive term series $\Sigma|u_n|$ formed by the moduli of the terms of the series is convergent.

5. Conditional convergent

A series is said to be conditionally convergent if it is convergent without being absolutely convergent.

Theorem: Every absolute convergent series is convergent.

Note. (i) If Σu_n is cgt without being absolutely cgt. I.e. if Σu_n is conditionally cgt then each of the +ve term series $\Sigma g(n)$ and $\Sigma h(n)$ diverges to infinity which follows from

$$g(n) = \frac{1}{2}\left[|u_n| + u_n\right]$$

$$h(n) = \frac{1}{2}\left[|u_n| - u_n\right]$$

(ii) It should be noted that three are no comparison tests for the cgt of conditionally cgt series.

Alternating series

A series whose terms are alternately +ve and –ve is called an alternating series

6. Leibnitz's test

Let u be a sequence such that $\forall n \in N$

(i) $u_n \geq 0$

(ii) $u_{n+1} \leq u_n$

(iii) lim u = 0

Then alternating series u(1) – u(2) + u(3) – u(4) + ... + $(-1)^{n+1}$ u(n) ..... is cgt.

7. Abel's Test

If a_n is a positive, monotonic decreasing function and if Σu_n is convergent series, then the series $\Sigma u_n a_n$ is also convergent.

Uniform convergence

Point wise Convergence of Sequence of Functions

Definition: A sequence of functions $\{f_n\}$ defined on [a, b] is said to be point-wise convergent to a function f on [a, b], if

to each $\in > 0$ to each $x \in [a, b]$, there exists a positive integer m (depending on ε and the point x) such that

$$|f_n(x) - f(x)| < \varepsilon \ \forall\ n > m \text{ and } \forall\ x \in [a,b].$$

The function f is called the point-wise limit of the sequence $\{f_n\}$. We write $\lim_{n\to\infty} f_n(x) = f(x)$.

5. FOURIER SERIES

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\alpha} a_n \cos nx + \sum_{n=1}^{n} b_n \sin nx$$

Where $(0 < x < 2\pi)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$$

And $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$$

And for $(-\pi < x < \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

And $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Where f(x) is an odd function; $a_0 = 0$ and $a_n = 0$ where f(x) is an even function; $b_n = 0$.

Fourier series in the interval $(0 < x < 2\ell)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Where $a_0 = \frac{1}{l}\int_0^{2l} f(x)dx$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}dx$$

And $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx$

In the interval $(-\ell < x < \ell)$

$$a_0 = \frac{1}{l}\int_{-l}^{+l} f(x)\,dx,\; a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}dx$$

And $b_n = \frac{1}{l}\int_{-l}^{+l} f(x)\sin\frac{n\pi x}{l}\,dx$

Note: When f(x) is an odd function, $a_0 = 0$ and $a_n = 0$ when f(x) is an even function, $b_n = 0$.

Half-Range series ($0 < x < \pi$)

A function f(x) defined in the interval $0 < x < \pi$ has two distinct half-range series.

(i) The half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx$$

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ and $a_n = \int_0^{\pi} f(x)\cos nx\,dx$

(ii) The half range sine series is,

$$f(x) = \Sigma b_n \sin nx$$

Where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$

Half-Range Series ($0 < x < l$)

A function f (x) defined in the interval $(0 < x < l)$ and having two distinct half-range series.

(i) The half range cosine series is,

$$f(x) = \frac{a_0}{2} + \Sigma a_n \cos\frac{n\pi x}{l}$$

Where $a_0 = \frac{2}{l}\int_0^{l} f(x)dx$

And $a_n = \frac{2}{l}\int_0^{l} f(x)\frac{\cos n\pi x}{l}dx$

(ii) The half-range sine series is,

$$f(x) = \Sigma b_n \sin \frac{n\pi x}{l}$$

Where $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx$

Complex form of Fourier Series

$$f(x) = \sum_{m=-\infty}^{+\infty} c_m e^{imx}$$

Where $c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx} dx$

$$c_0 = \int_{-\pi}^{+\pi} f(x) dx \text{ and}$$

$$C_{-m} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x) e^{imx} dx.$$

Parseval's Identity

For Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, 0 < x < 2l$$

The Parseval's identity is

$$\frac{1}{2l}\int_0^{2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

FOURIER INTEGRAL

The Fourier series of periodic function f (x) on the interval $(-\ell, +\ell)$ is given by

$$f(x) = a_0 + \frac{n\pi x}{\ell} \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell} \quad .....(1)$$

Where $a_0 = \frac{1}{2\ell}\int_{-\ell}^{+\ell} f(x)\, dx = \frac{1}{2\ell}\int_{-\ell}^{+\ell} f(t)\, dt$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{+\ell} f(t)\cos\frac{n\pi t}{\ell} dt$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{+\ell} f(t)\sin\frac{n\pi t}{\ell} dt$$

Then

$$f(x) = \frac{1}{\pi}\int_0^{\infty} du \int_{-\infty}^{+\infty} f(t)\cos u(x-t)dt$$

This is a form of Fourier Integral.

SOME PROBLEMS

1. The set of all positive values of a for which the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\tan^{-1}\left(\frac{1}{n}\right)\right)^a$ converges, is

(A) $\left(0,\frac{1}{3}\right]$ (B) $\left(0,\frac{1}{3}\right)$ (C) $\left[\frac{1}{3},\infty\right)$ (D) $\left(\frac{1}{3},\infty\right)$

2. Match the following

Series (X)	Domain of convergence (Y)
A. $\sum \frac{x^n}{n^3}$	(i) [0, 2]
B. $\sum (-1)^n \frac{x^{2n+1}}{2n+1}$	(ii) [–2 –e, –2 + e]
C. $\sum \frac{(-1)^{n+1}}{n}(x-1)^n$	(iii) [–1, 1]
D. $\sum \frac{n!(x+2)^n}{n^n}$	(iv)]–1, 1[

	A	B	C	D
(A)	(iv)	(iii)	(ii)	(i)
(B)	(iv)	(iii)	(i)	(ii)
(C)	(iii)	(iv)	(i)	(ii)
(D)	(i)	(ii)	(iv)	(iii)

3. The series

$$1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \ldots \text{ is –}$$

(A) Convergent, if $p \geq 2$ divergent, if $p < 2$

(B) Convergent, if $p > 2$ and divergent, if $p \leq 2$

(C) Convergent, if $p \leq 2$ and divergent, if $p > 2$

(D) Convergent, if $p < 2$ and divergent, if $p \geq 2$

4. For the improper integral $\int_0^1 x^{\alpha-1}e^{-x}dx$ which one of the following is true ?

(A) if $\alpha < 0$, convergent and if $\alpha = 0$, divergent

(B) if $\alpha \geq 0$, Convergent and if $\alpha < 0$, divergent

(C) if $\alpha > 0$, convergent and if $\alpha \leq 0$, divergent

(D) If $\alpha > 0$, divergent and if $\alpha \leq 0$, convergent

5. Let $A \subseteq R$ and Let $f_1 f_2 - f_n$ be functions on A to R and Let c be a cluster point of A if $L_k = \lim_{x\to c} f_k$ for k = 1, ..., n Then $\lim_{x\to c} [f(x)]^c$

(A) L (B) L_k $k \in N$ (C) L^n (D) 1

ANSWER KEY :- 1. (D) , 2. (B) , 3. (B) , 4. (C) , 5. (C)

1. (D) Use the following results:

(1) Let Σa_n & Σb_n be two positive term series

(i) If $\text{Lt}_{n\to\infty} \frac{a_n}{b_n} = \ell$, ℓ being a finite non–zero constant, then Σa_n & Σb_n both converge or diverge together.

(ii) If $\text{Lt}_{n\to\infty} \frac{a_n}{b_n} = 0$ & $\Sigma\beta\nu$ converges, then Σa_n also converges.

(2) The series $\sum \frac{1}{n^p}$ converges if p > 1 & diverges if $p \leq 1$. We compare the given series with the series $\sum \frac{1}{n^{ap}}$

$$\text{Lt}_{n\to\infty} \frac{\left(\frac{1}{n} - \tan^{-1}\frac{1}{n}\right)^a}{\frac{1}{n^{ap}}} = \text{Lt}_{n\to\infty} \frac{\left(\frac{1}{3n^3} - \frac{1}{5n^5}.......\right)^a}{\frac{1}{n^{pa}}} \left[\because \frac{1}{n} - \tan^{-1}\left(\frac{1}{n}\right) = \frac{1}{n} - \left[\frac{1}{n} - \frac{1}{3n^3} + ...\right]\right]$$

$$= \frac{1}{3n^3} - \frac{1}{5n^5} + ...$$

$$= \text{Lt}_{n\to\infty} \left(\frac{n^p}{3n^3} - \frac{n^p}{5n^5} - - - -\right)^a$$

For this limit to be zero or some other finite number

$3 - p \geq 0$ i.e. $p \leq 3$

& for the series $\sum \frac{1}{n^{ap}}$ to be convergent, ap > 1

$\Rightarrow \quad a > \frac{1}{p} \geq \frac{1}{3}$

$\Rightarrow \quad a > \frac{1}{3}$

$\Rightarrow \quad a \in \left(\frac{1}{3}, \infty\right)$ $\therefore$ **Ans.** is **(D)**

2. (B) **(i)** $\sum \frac{x^n}{n^3}$

$\because \ a_n = \frac{1}{n^3}; a_{n+1} = \frac{1}{(n+1)^3}$

$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^3 = 1$

So the domain of a_n is]–1, 1[$\sum \frac{1}{n^2}$

For x = 1 the given power series is

Which is convergent.

For x = –1 the given power series is

$-1 + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} \ldots$

Which is convergent, by leibnitz's test. $\therefore$ **Ans.** is (iv)

(ii) $\sum (-1)^n \frac{x^{2n+1}}{2n+1}$

$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{2n+3}{2n+1} = 1$

The interval of convergence [–1, 1]

for x = 1, the series becomes

$1 - \frac{1}{3} + \frac{1}{5} \ldots$ Which is convergent by Leibnitz's test

For x = –1 the series becomes $-1 + \frac{1}{3} - \frac{1}{5} \ldots$

Which is again convergent.

Hence the exact interval of convergency is [–1, 1].   ∴ **Ans.** is (iii)

(iii) $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n}{n-1}\right| = 1$

Since the given power series is about the point x = 1 the interval of convergence is

–1 + 1 < x < 1 + 1 = 0 < x < 2

for x = +2, the given series $\sum\frac{(-1)^{n+1}}{n}$ which is convergent by leibnitz's test.

Hence the exact interval of convergence is [0, 2].   ∴ **Ans.** is (i)

(iv) $\sum\frac{n!(x+2)^n}{n^n}$

The given power series is about the point x = 2

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{n!}{n^n}\cdot\frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$ ∴ **Ans.** is (ii)

The interval of convergence is [–2 –e, –2+ e],

3. (B) Neglecting the first term

$$u_n = \left(\frac{1.3.5.....(2n-1)}{2.4.6.....2n}\right)^p$$

and $u_{n+1} = \left(\frac{1.3.5.....(2n-1)(2n+1)}{2.4.6.....(2n)(2n+2)}\right)^p$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p}$$

or, $\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p} = 1$

∴ Ratio test fails.

$$\therefore \log\frac{u_n}{u_{n+1}} = \log\left\{\frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p}\right\}$$

$$= p\log\left(1+\frac{1}{n}\right) - p\log\left(1+\frac{1}{2n}\right)$$

$$= p\left[\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)-\left(\frac{1}{2n}-\frac{1}{8n^2}+\frac{1}{24n^3}\right)\right]$$

$$= p\left[\left(\frac{1}{n}-\frac{1}{2n^2}\right)-\left(\frac{1}{2n}-\frac{1}{8n^2}\right)+\left(\frac{1}{3n^3}-\frac{1}{24n^3}\right)+\ldots\right]$$

$$= p\left[\frac{1}{2n}-\frac{3}{8n^2}+\frac{7}{24n^3}+\ldots.\right]$$

$$\therefore \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}}$$

$$= \lim_{n\to\infty} p\left(\frac{1}{2}-\frac{3}{8n}+\frac{7}{24n^2}+\ldots.\right)$$

$$= \frac{p}{2}$$

From Logarithmic test.

The series is convergent, if $\frac{1}{2}p > 1$, i.e., $p > 2$

The series is divergent, if $\frac{1}{2}p < 1$, i.e., $p < 2$

The test fails, if $\frac{1}{2}p = 1$ i.e., $p = 2$

Now $n\log\frac{u_n}{u_{n+1}} = 2\left(\frac{1}{2}-\frac{3}{8n}+\frac{7}{24n^2}+\ldots\right)$

or, $\left\{n\log\frac{u_n}{u_{n+1}}-1\right\}$

$$= \left\{\left(1-\frac{3}{4n}+\frac{7}{12n^2}+\ldots\right)-1\right\}$$

$$= -\frac{3}{4n}+\frac{7}{12n^2}+\ldots..$$

or, $\left\{n\log\frac{u_n}{u_{n+1}}-1\right\}\log n$

$$= -\frac{3}{4}\times\frac{\log n}{n} + \frac{7}{12}\times\frac{\log n}{n^2} + \ldots.$$

or, $\lim_{n\to\infty}\left(-\frac{3}{4}\times\frac{\log n}{n}+\frac{7}{12}\times\frac{\log n}{n^2}....\right)$

Hence by higher logarithmic test the given series is divergent, if p = 2.

Hence the given series is convergent when p > 2 and divergent when $p \le 2$.

The correct answer is (2).

4. (C) $\int_0^1 x^{\alpha-1}e^{-x}dx,$

When $\alpha > 1$, the given integral is a proper integral and hence it is convergent. When $\alpha < 1$, the integrand becomes infinite at x = 0.

Now $\lim_{x\to 0} x^{\mu}.x^{\alpha-1}e^{-x} = \lim_{x\to 0} x^{\mu+\alpha-1}e^{-x} = 1$

if $\mu+\alpha-1=0$, i.e., $\mu = 1-\alpha$

We then have $0 < \mu < 1$ when $0 < \alpha < 1$

and $\mu \ge 1$ where $\alpha \le 0$.

It follows by μ -test that the integral is convergent when $0 < \alpha < 1$ and divergent when $\alpha \le 0$.

And we have proved above that the integral is convergent when $\alpha \ge 1$. Consequently the given integral is convergent if $\alpha > 0$ and divergent if $\alpha \le 0$.

5. (C) if $L_k = \lim_{x\to c} f_k$

then it follows from a by known result which is called an Induction argument that

$$L_1 + L_2 + \cdots + L_n = \lim_{x\to c} f(_1 + f_2 + \cdots + f_n),$$

and

$$L_1 \cdot L_2 \cdots L_n = \lim(f_1 \cdot f_2 \cdots f_n).$$

In particular, we deduce that if $L = \lim_{x\to c} f$ and $n \in N$, then

$$L^n = \lim_{x\to c} (f(x))^n.$$