



CSIR NET - MATHEMATICAL SCIENCE

SAMPLE THEORY

SEQUENCES, SERIES AND LIMIT POINTS OF SEQUENCES

- SEQUENCES
- LIMITS : INFERIOR & SUPERIOR
- ALGEBRA OF SEQUENCES
- SEQUENCE TESTS
- FOURIER SERIES
- SOME PROBLEMS

VPM CLASSES

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SEQUENCE

A sequence in a set S is a function whose domain is the set N of natural numbers and whose range is a subset of S . A sequence whose range is a subset of R is called a real sequence.

$$S_n = u_1 + u_2 + u_3 + \dots + u_n$$

$$S_1 = u_1$$

$$S_2 = u_1 + u_2$$

$$S_3 = u_1 + u_2 + u_3$$

...

...

$$S_n = u_1 + u_2 + u_3 + \dots + u_n \rightarrow \text{series}$$

↓

Sequence

Bounded Sequence: A sequence is said to be bounded if and only if its range is bounded. Thus a sequence S_n is bounded if there exists

$$k \leq S_n \leq K, \forall n \in N$$

$$\Leftrightarrow S_n \in [k, K]$$

The l. u. b (Supremum) and the g.l.b (infimum) of the range of a bounded sequence may be referred as its g.l.b and l.u.b respectively.

Limits inferior and Superior:

From the definition of limit in Section 1.4, it follows that the limiting behavior of any sequence $\{a_n\}$ of real numbers depends only on sets of the form $\{a_n : n \geq m\}$, i.e., $\{a_m, a_{m+1}, a_{m+2}, \dots\}$. In this regard we make the following definition.

Definition: Let $\{a_n\}$ be a sequence of real numbers (not necessarily bounded). We define

$$\liminf_{n \rightarrow \infty} a_n = \sup_n \inf \{a_n, a_{n+1}, a_{n+2}, \dots\}$$

And $\limsup_{n \rightarrow \infty} a_n = \inf_n \sup \{a_n, a_{n+1}, a_{n+2}, \dots\}$

As the limit inferior and limit superior respectively of the sequence $\{a_n\}$.

We shall denote limit inferior and limit superior of $\{a_n\}$ by $\liminf_{n \rightarrow \infty} a_n$ and $\limsup_{n \rightarrow \infty} a_n$ or simply by $\lim a_n$ and $\overline{\lim} a_n$ respectively.

We shall use the following notations for the sequence $\{a_n\}$, for each $n \in \mathbf{N}$

$$\underline{A}_n = \inf \{a_n, a_{n+1}, a_{n+2}, \dots\},$$

And $\overline{A}_n = \sup \{a_n, a_{n+1}, a_{n+2}, \dots\}$.

Therefore, we have

$$\liminf a_n = \sup_n \underline{A}_n$$

And $\overline{\lim} a_n = \inf_n \overline{A}_n$

Now $\{a_{n+1}, a_{n+2}, \dots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \dots\}$, Therefore by taking infimum and supremum respectively, it follows that

$$\underline{A}_{n+1} \geq \underline{A}_n \text{ And } \overline{A}_{n+1} \leq \overline{A}_n$$

This is true for each $n \in \mathbf{N}$.

The above inequalities show that the associated sequences $\{\underline{A}_n\}$ and $\{\overline{A}_n\}$ monotonically increase and decrease respectively with n .

Remark: It should be noted that both limits inferior and superior exist uniquely (finite or infinite) for all real sequences.

Theorem: If $\{a_n\}$ is any sequence, then

$$\liminf (-a_n) = -\overline{\lim} a_n, \text{ and } \overline{\lim} (-a_n) = -\liminf a_n.$$

Let $b_n = -a_n$, $n \in \mathbf{N}$ then we have

$$\begin{aligned} \underline{B}_n &= \inf \{b_n, b_{n+1}, \dots\} \\ &= -\sup \{a_n, a_{n+1}, \dots\} = -\overline{A}_n \end{aligned}$$

And so

$$\liminf (-a_n) = \liminf b_n = \sup \{\underline{B}_1, \underline{B}_2, \dots\}$$

$$= \sup \{-\bar{A}_1, -\bar{A}_2, \dots\}$$

$$= - \inf \{\bar{A}_1, \bar{A}_2, \dots\}$$

$$= - \inf \bar{A}_n = - \overline{\lim} a_n.$$

Also, $\underline{\lim} a_n = \underline{\lim} (-a_n) = - \overline{\lim} (-a_n).$

Theorem: If $\{a_n\}$ is any sequence, then

$$\underline{\lim} a_n = -\infty \text{ if and only if } \{a_n\} \text{ is not bounded below,}$$

And $\overline{\lim} a_n = +\infty$ if and only if $\{a_n\}$ is not bounded above.

Let $\underline{A}_n = \inf \{a_n, a_{n+1}, \dots\},$

And $\bar{A}_n = \sup \{a_n, a_{n+1}, \dots\}, n \in \mathbf{N}$

By definition we have

$$\underline{\lim} a_n = -\infty \Leftrightarrow \sup \{\underline{A}_1, \underline{A}_2, \dots\} = -\infty$$

$$\Leftrightarrow \underline{A}_n = -\infty, \quad \forall n \in \mathbf{N}$$

$$\Leftrightarrow \inf \{a_n, a_{n+1}, \dots\} = -\infty, \quad \forall n \in \mathbf{N}$$

$$\Leftrightarrow \{a_n\} \text{ is not bounded below:}$$

The proof for limit superior is similar.

Corollary: If $\{a_n\}$ is any sequence, then

(i) $-\infty < \underline{\lim} a_n \leq +\infty$ iff $\{a_n\}$ is bounded below.

and

(ii) $-\infty \leq \overline{\lim} a_n < +\infty$ iff $\{a_n\}$ is bounded above.

For bounded sequences, we have the following useful criteria for limits inferior and superior respectively.

Limit points of a sequence:

A number ξ is said to be a limit point of a sequence S_n if given any rbd of ξ , S_n belongs to the same for an infinite number of values of n .

Now $\{S_{n+1}, S_{n+2}, S_{n+3}, \dots\} \subseteq \{S_n, S_{n+1}, S_{n+2}, \dots\}$, therefore by taking infimum and supremum respectively, it follows that $\underline{A}_{n+1} \geq \underline{A}_n$ and $\overline{A}_{n+1} \leq \overline{A}_n$ for each $n \in \mathbb{N}$

Remark: Both limits inferior and superior exist uniquely (finite or infinite) for all real sequence.

Theorem: If $\{S_n\}$ is any sequence, then

$$\inf S_n \leq \liminf S_n \leq \sup S_n$$

If $\{S_n\}$ is any sequence, then

$$\liminf \{-S_n\} = -\overline{\lim} S_n$$

$$\text{And } -\overline{\lim} \{-S_n\} = \overline{\lim} S_n$$

Some Important Properties of Algebra of sequences

1. If $\{a_n\}$ is a bounded sequence such that $a_n > 0$ for all $n \in \mathbb{N}$, then

$$(i) \liminf \left(\frac{1}{a_n} \right) = \frac{1}{\overline{\lim} a_n}, \text{ if } \overline{\lim} a_n > 0$$

$$(ii) \limsup \left(\frac{1}{a_n} \right) = \frac{1}{\liminf a_n}, \text{ if } \liminf a_n > 0$$

5. If $\{a_n\}$ and $\{b_n\}$ are bounded sequence, $a_n \geq 0, b_n > 0$ for all $n \in \mathbb{N}$, then

$$(i) \liminf \left(\frac{a_n}{b_n} \right) \geq \frac{\liminf a_n}{\overline{\lim} b_n}, \text{ if } \overline{\lim} b_n > 0$$

$$(ii) \overline{\lim} \left(\frac{a_n}{b_n} \right) \leq \frac{\overline{\lim} a_n}{\liminf b_n}, \text{ if } \liminf b_n > 0$$

SOME IMPORTANT SEQUENCE TESTS

1. **Cauchy's root test**

Let $\sum u_n$ be +ve term series and

$$\lim_{n \rightarrow \infty} \{u_n\}^{1/n} = \ell$$

Then the series is

- (i) Cgt if $\ell < 1$
- (ii) Dgt if $\ell > 1$
- (iii) No firm decision is possible if $\ell = 1$

2. Raabe's test

Let $\sum u_n$ be a +ve term series and

$$\lim_n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} = \ell$$

then the series is

- (i) Cgt if $\ell > 1$
- (ii) Dgt if $\ell < 1$
- (iii) No firm decision is possible if $\ell = 1$

3. Logarithmic Test:

If $\sum u_n$ is +ve terms series such that

$$\lim_{n \rightarrow \infty} \left(n \log \frac{u_n}{u_{n+1}} \right) = \ell$$

Then the series

- (i) cgt if $\ell > 1$
- (ii) dgt if $\ell < 1$

4. Absolute convergent

A series $\sum u_n$ is said to be absolutely cgt if the positive term series $\sum |u_n|$ formed by the module of the terms of the series is convergent.

5. Conditional convergent

A series is said to be conditionally convergent if it is convergent without being absolutely convergent.

Theorem: Every absolute convergent series is convergent.

Note. (i) If $\sum u_n$ is cgt without being absolutely cgt. I.e. if $\sum u_n$ is conditionally cgt then each of the +ve term series $\sum g(n)$ and $\sum h(n)$ diverges to infinity which follows from

$$g(n) = \frac{1}{2} [|u_n| + u_n]$$

$$h(n) = \frac{1}{2} [|u_n| - u_n]$$

(ii) It should be noted that there are no comparison tests for the cgt of conditionally cgt series

Alternating series

A series whose terms are alternately +ve and -ve is called an alternating series.

6. Leibnitz's test

Let u be a sequence such that $\forall n \in \mathbb{N}$

(i) $u_n \geq 0$

(ii) $u_{n+1} \leq u_n$

(iii) $\lim u = 0$

Then alternating series $u(1) - u(2) + u(3) - u(4) + \dots + (-1)^{n+1} u(n) \dots$ is cgt.

7. Abel's Test

If a_n is a positive, monotonic decreasing function and if $\sum u_n$ is convergent series, then the series $\sum u_n a_n$ is also convergent.

Uniform convergence

Point wise Convergence of Sequence of Functions

Definition: A sequence of functions $\{f_n\}$ defined on $[a, b]$ is said to be point-wise convergent to a function f on $[a, b]$, if

to each $\epsilon > 0$ to each $x \in [a, b]$, there exists a positive integer m (depending on ϵ and the point x) such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall n > m \text{ and } \forall x \in [a, b].$$

The function f is called the point-wise limit of the sequence $\{f_n\}$. We write $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

FOURIER SERIES

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $(0 < x < 2\pi)$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx$$

And $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx$

And for $(-\pi < x < \pi)$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

And $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$

Where $f(x)$ is an odd function; $a_0 = 0$ and $a_n = 0$ where $f(x)$ is an even function; $b_n = 0$.

Fourier series in the interval $(0 < x < 2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Where $a_0 = \frac{1}{l} \int_0^{2l} f(x) dx$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx$$

And $b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx$

In the interval $(-\ell < x < \ell)$

$$a_0 = \frac{1}{l} \int_{-l}^{+l} f(x) dx, \quad a_n = \frac{1}{l} \int_{-l}^{+l} f(x) \cos \frac{n\pi x}{l} dx$$

$$\text{And } b_n = \frac{1}{l} \int_{-l}^{+l} f(x) \sin \frac{n\pi x}{l} dx$$

Note: When $f(x)$ is an odd function, $a_0 = 0$ and $a_n = 0$ when $f(x)$ is an even function, $b_n = 0$.

Half-Range series ($0 < x < \pi$)

A function $f(x)$ defined in the interval $0 < x < \pi$ has two distinct half-range series.

(i) The half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx$$

$$\text{Where } a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx \quad \text{and} \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

(ii) The half range sine series is,

$$f(x) = \sum b_n \sin nx$$

$$\text{Where } b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

Half-Range Series ($0 < x < l$)

A function $f(x)$ defined in the interval ($0 < x < l$) and having two distinct half-range series.

(i) The half range cosine series is,

$$f(x) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{l}$$

$$\text{Where } a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$\text{And } a_n = \frac{2}{l} \int_0^l f(x) \frac{\cos n\pi x}{l} dx$$

(ii) The half-range sine series is,

$$f(x) = \sum b_n \sin \frac{n\pi x}{l}$$

$$\text{Where } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

Complex form of Fourier series

$$f(x) = \sum_{m=-\infty}^{+\infty} c_m e^{imx}$$

$$\text{Where } c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx$$

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \text{ and}$$

$$c_{-m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{imx} dx.$$

Parseval's Identity

For Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, 0 < x < 2l$$

The Parseval's identity is

$$\frac{1}{2l} \int_0^{2l} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

FOURIER INTEGRAL

The Fourier series of periodic function $f(x)$ on the interval $(-l, +l)$ is given by

$$f(x) = a_0 + \frac{n\pi x}{l} \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \dots(1)$$

$$\text{Where } a_0 = \frac{1}{2l} \int_{-l}^{+l} f(x) dx = \frac{1}{2l} \int_{-l}^{+l} f(t) dt$$

$$a_n = \frac{1}{l} \int_{-l}^{+l} f(t) \cos \frac{n\pi t}{l} dt$$

$$b_n = \frac{1}{l} \int_{-l}^{+l} f(t) \sin \frac{n\pi t}{l} dt$$

Then

$$f(x) = \frac{1}{\pi} \int_0^{\infty} du \int_{-\infty}^{+\infty} f(t) \cos u(x-t) dt$$

This is a form of Fourier Integral.

SOME PROBLEMS

1. The set of all positive values of a for which the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \tan^{-1} \left(\frac{1}{n} \right) \right)^a$ converges, is

(1) $\left(0, \frac{1}{3} \right)$ (2) $\left(0, \frac{1}{3} \right]$ (3) $\left[\frac{1}{3}, \infty \right)$ (4) $\left[\frac{1}{3}, \infty \right]$

2. Match the following

Series (X)

Domain of
Convergence (Y)

A. $\sum \frac{x^n}{n^3}$

(i) $[0, 2]$

B. $\sum (-1)^n \frac{x^{2n+1}}{2n+1}$

(ii) $[-2-e, -2+e]$

C. $\sum \frac{(-1)^{n+1}}{n} (x-1)^n$

(iii) $[-1, 1]$

D. $\sum \frac{n!(x+2)^n}{n^n}$

(iv) $[-1, 1]$

	A	B	C	D
(1)	(iv)	(iii)	(ii)	(i)
(2)	(iv)	(ii)	(i)	(ii)
(3)	(iii)	(iv)	(i)	(ii)
(4)	(i)	(ii)	(iv)	(iii)

3. The series

$$1^p + \left(\frac{1}{2} \right)^p + \left(\frac{1.3}{2.4} \right)^p + \left(\frac{1.3.5}{2.4.6} \right)^p + \dots \text{ is -}$$

- (1) Convergent, if $p \geq 2$ divergent, if $p < 2$
 (2) Convergent, if $p > 2$ and divergent, if $p \leq 2$

(3) Convergent, if $p \leq 2$ and divergent, if $p > 2$

(4) Convergent, if $p < 2$ and divergent, if $p \geq 2$

4. For the improper integral $\int_0^1 x^{\alpha-1} e^{-x} dx$ which one of the following is true

(1) if $\alpha < 0$, convergent and if $\alpha = 0$, divergent

(2) if $\alpha > 0$, Convergent and if $\alpha < 0$, divergent

(3) if $\alpha > 0$, convergent and if $\alpha \leq 0$, divergent

(4) If $\alpha > 0$, divergent and if $\alpha \leq 0$, convergent

5. Let $A \subseteq \mathbb{R}$ and Let f_1, f_2, \dots, f_n be functions on A to \mathbb{R} and Let c be a cluster point of A if $L_k = \lim_{x \rightarrow c} f_k$ for $k = 1, \dots, n$. Then $\lim_{x \rightarrow c} [f(x)]^c$

(1) L

(2) $L_k, k \in \mathbb{N}$

(3) L^n

(4) 1

ANSWER KEY: - 1. (4), 2. (2), 3. (2), 4. (3), 5. (3)

1. (4) Use the following results:

(1) Let $\sum a_n$ & $\sum b_n$ be two positive term series

(i) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \ell$, ℓ being a finite non-zero constant, then $\sum a_n$ & $\sum b_n$ both converge or diverge together.

(ii) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ & $\sum b_n$ converges, then $\sum a_n$ also converges.

(2) The series $\sum \frac{1}{n^p}$ converges if $p > 1$ & diverges if $p \leq 1$. We compare the given series with the

series $\sum \frac{1}{n^{ap}}$

$$\lim_{n \rightarrow \infty} \frac{\left(\frac{1}{n} - \tan^{-1} \frac{1}{n}\right)^a}{\frac{1}{n^{ap}}} = \lim_{n \rightarrow \infty} \frac{\left(\frac{1}{3n^3} - \frac{1}{5n^5} + \dots\right)^a}{\frac{1}{n^{pa}}} \left[\because \frac{1}{n} - \tan^{-1} \left(\frac{1}{n}\right) = \frac{1}{n} - \left[\frac{1}{n} - \frac{1}{3n^3} + \dots\right] \right]$$

$$= \frac{1}{3n^3} - \frac{1}{5n^5} + \dots$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n^p}{3n^3} - \frac{n^p}{5n^5} \right)^a$$

For this limit to be zero or some other finite number

$$3 - p \geq 0 \quad \text{i.e. } p \leq 3$$

& for the series $\sum \frac{1}{n^{ap}}$ to be convergent, $ap > 1$

$$\Rightarrow a > \frac{1}{p} \geq \frac{1}{3}$$

$$\Rightarrow a > \frac{1}{3}$$

$$\Rightarrow a \in \left(\frac{1}{3}, \infty \right) \quad \therefore \text{Ans. is (D)}$$

2. (2) (i) $\sum \frac{x^n}{n^3}$

$$\therefore a_n = \frac{1}{n^3}; a_{n+1} = \frac{1}{(n+1)^3}$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^3 = 1$$

So the domain of a_n is $[-1, 1] \sum \frac{1}{n^2}$

For $x = 1$ the given power series is

Which is convergent.

For $x = -1$ the given power series is

$$-1 + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} \dots$$

Which is convergent, by Leibnitz's test.

\therefore Ans. is (iv)

(ii) $\sum (-1)^n \frac{x^{2n+1}}{2n+1}$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{2n+3}{2n+1} = 1$$

The interval of convergence $[-1, 1]$

for $x = 1$, the series becomes

$$1 - \frac{1}{3} + \frac{1}{5} \dots \text{ Which is convergent by Leibnitz's test}$$

For $x = -1$ the series becomes $-1 + \frac{1}{3} - \frac{1}{5} \dots$

Which is again convergent.

Hence the exact interval of convergence is $[-1, 1]$. \therefore Ans. is (iii)

$$(iii) R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n-1} \right| = 1$$

Since the given power series is about the point $x = 1$ the interval of convergence is $-1 + 1 < x < 1 + 1 = 0 < x < 2$

for $x = +2$, the given series $\sum \frac{(-1)^{n+1}}{n}$ which is convergent by Leibnitz's test.

Hence the exact interval of convergence is $[0, 2]$. \therefore Ans. is (i)

$$(iv) \sum \frac{n!(x+2)^n}{n^n}$$

The given power series is about the point $x = 2$

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \rightarrow \infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e \quad \therefore \text{Ans. is (ii)}$$

The interval of convergence is $[-2 - e, -2 + e]$,

3. (2) Neglecting the first term

$$u_n = \left(\frac{1.3.5 \dots (2n-1)}{2.4.6 \dots 2n} \right)^p$$

$$\text{and } u_{n+1} = \left(\frac{1.3.5 \dots (2n-1)(2n+1)}{2.4.6 \dots (2n)(2n+2)} \right)^p$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)^p}{(2n+1)^p} = \frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p}$$

$$\text{or, } \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p} = 1$$

\therefore Ratio test fails.

$$\therefore \log \frac{u_n}{u_{n+1}} = \log \left\{ \frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p} \right\}$$

$$= p \log \left(1+\frac{1}{n}\right) - p \log \left(1+\frac{1}{2n}\right)$$

$$= p \left[\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots \right) - \left(\frac{1}{2n} - \frac{1}{8n^2} + \frac{1}{24n^3} \right) \right]$$

$$= p \left[\left(\frac{1}{n} - \frac{1}{2n^2} \right) - \left(\frac{1}{2n} - \frac{1}{8n^2} \right) + \left(\frac{1}{3n^3} - \frac{1}{24n^3} \right) + \dots \right]$$

$$= p \left[\frac{1}{2n} - \frac{3}{8n^2} + \frac{7}{24n^3} + \dots \right]$$

$$\therefore \lim_{n \rightarrow \infty} n \log \frac{u_n}{u_{n+1}}$$

$$= \lim_{n \rightarrow \infty} p \left(\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} + \dots \right)$$

$$= \frac{p}{2}$$

From Logarithmic test.

The series is convergent, if $\frac{1}{2}p > 1$, i.e., $p > 2$

The series is divergent, if $\frac{1}{2}p < 1$, i.e., $p < 2$

The test fails if $\frac{1}{2}p = 1$ i.e., $p = 2$

$$\text{Now } n \log \frac{u_n}{u_{n+1}} = 2 \left(\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} + \dots \right)$$

$$\begin{aligned} \text{or, } \left\{ n \log \frac{u_n}{u_{n+1}} - 1 \right\} \\ &= \left\{ \left(1 - \frac{3}{4n} + \frac{7}{12n^2} + \dots \right) - 1 \right\} \\ &= -\frac{3}{4n} + \frac{7}{12n^2} + \dots \end{aligned}$$

$$\begin{aligned} \text{or, } \left\{ n \log \frac{u_n}{u_{n+1}} - 1 \right\} \log n \\ &= -\frac{3}{4} \times \frac{\log n}{n} + \frac{7}{12} \times \frac{\log n}{n^2} + \dots \end{aligned}$$

$$\text{or, } \lim_{n \rightarrow \infty} \left(-\frac{3}{4} \times \frac{\log n}{n} + \frac{7}{12} \times \frac{\log n}{n^2} + \dots \right)$$

Hence by higher logarithmic test the given series is divergent if $p = 2$.

Hence the given series is convergent when $p > 2$ and divergent when $p \leq 2$.

The correct answer is (2).

4. (3) $\int_0^1 x^{\alpha-1} e^{-x} dx,$

When $\alpha > 1$, the given integral is a proper integral and hence it is convergent. When $\alpha < 1$, the integrand becomes infinite at $x = 0$.

$$\text{Now } \lim_{x \rightarrow 0} x^\mu \cdot x^{\alpha-1} e^{-x} = \lim_{x \rightarrow 0} x^{\mu+\alpha-1} e^{-x} = 1$$

$$\text{if } \mu + \alpha - 1 = 0, \text{ i.e., } \mu = 1 - \alpha$$

We then have $0 < \mu < 1$ when $0 < \alpha < 1$

and $\mu \geq 1$ where $\alpha \leq 0$.

It follows by μ -test that the integral is convergent when $0 < \alpha < 1$ and divergent when $\alpha \leq 0$.

And we have proved above that the integral is convergent when $\alpha \geq 1$. Consequently the given integral is convergent if $\alpha > 0$ and divergent if $\alpha \leq 0$.

5. (3) if $L_k = \lim_{x \rightarrow c} f_k$

then it follows from a by known result which is called an Induction argument that

$$L_1 + L_2 + \dots + L_n = \lim_{x \rightarrow c} (f_1 + f_2 + \dots + f_n),$$

and

$$L_1 \cdot L_2 \cdots L_n = \lim_{x \rightarrow c} (f_1 \cdot f_2 \cdots f_n).$$

In particular, we deduce that if $L = \lim_{x \rightarrow c} f$ and $n \in \mathbb{N}$, then

$$L^n = \lim_{x \rightarrow c} (f(x))^n.$$

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